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RICCI FLOW AND THE UNIFORMIZATION ON COMPLETE NONCOMPACT KÄHLER MANIFOLDS

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1. Introduction

In the theory of complex geometry, the complete Kähler manifolds with positive holomorphic bisectional curvature have been studied for many years. If M is a complete compact Kähler manifold of complex dimension n with positive holomorphic bisectional curvature, people conjectured that M is biholomorphic to $\mathbb{CP}^n$. This was the famous Frankel Conjecture and was solved by Mori [34] and Siu–Yau [46] in 1979. In the case where M is noncompact, Greene–Wu [18] and Yau have the following conjecture:

Conjecture. *Suppose M is a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then M is biholomorphic to $\mathbb{C}^n$.*

Several results concerning this conjecture were obtained in the past few years. In 1981, N. Mok, Y.T. Siu and S.T. Yau [31] proved the following theorem:

Theorem. *Suppose M is a complete noncompact Kähler manifold of complex dimension $n \geq 2$ with bounded and nonnegative holomorphic bisectional curvature. Suppose M is a Stein manifold. Suppose there exist constants $0 < \varepsilon, C_0, C_1 < +\infty$ such that*

(i) $\operatorname{Vol}(B(x_0,\gamma)) \geq C_0\gamma^{2n}$, $0 \leq \gamma < +\infty$,

(ii) $0 \leq R(x) \leq \dfrac{C_1}{\gamma(x,x_0)^{2+\varepsilon}}$, $x \in M$,

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where $B(x_0,\gamma)$ denotes the geodesic ball of radius γ and centered at x_0, Vol$(B(x_0,\gamma))$ denotes the volume of $B(x_0,\gamma)$, $R(x)$ denotes the scalar curvature, and $\gamma(x,x_0)$ denotes the distance between x and x_0. Then M is isometrically biholomorphic to $\mathbb{C}^n$ with the flat metric.

The method used in Mok–Siu–Yau's paper [31] is the study of the Poincarè–Lelong equation on complete noncompact Kähler manifolds. Their result was improved by N. Mok [32] in 1984. In his paper [32] Mok used some algebraic geometrical techniques to control the holomorphic functions of polynomial growth on M and obtained the following result:

Theorem. *Suppose M is a complete noncompact Kähler manifold of complex dimension n with bounded and positive holomorphic bisectional curvature. Suppose there exist constants $0 < C_0,\ C_1 < +\infty$ such that*

(i) $\mathrm{Vol}(B(x_0,\gamma)) \geq C_0\gamma^{2n},\ \ 0 \leq \gamma < +\infty,$

(ii) $0 < R(x) \leq \dfrac{C_1}{\gamma(x,x_0)^2},\ \ x \in M.$

Then M is biholomorphic to an affine algebraic variety.

Under the direction of S.T. Yau, the author of this paper proved the following result in his Ph.D. thesis [43] in 1990:

Theorem 1.1. *Suppose M is a complete noncompact Kähler manifold of complex dimension n with bounded and positive holomorphic bisectional curvature. Suppose there exist constants $0 < C_0,\ C_1 < +\infty$ such that*

(i) $\mathrm{Vol}(B(x_0,\gamma)) \geq C_0\gamma^{2n},\ \ 0 \leq \gamma < +\infty,$

(ii) $\displaystyle\int_{B(x_0,\gamma)} R(x)dx \leq \frac{C_1}{\gamma^2}\cdot \mathrm{Vol}(B(x_0,\gamma)),\ \ x_0 \in M,\ \ 0 \leq \gamma < +\infty.$

Then M is biholomorphic to $\mathbb{C}^n$.

The method which we used in [43] to prove Theorem 1.1 is the study of the following Ricci flow evolution equation of the metric on M:

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}(t), \tag{1}$$

where $g_{ij}(t)$ is a family of metrics, and $R_{ij}(t)$ denotes the Ricci curvature of $g_{ij}(t)$. Evolution equation (1) was originally developed by R.S. Hamilton [22] in 1982. Using evolution equation (1) Hamilton proved [22] that one can deform the metric on compact three-dimensional Riemannian manifolds with positive Ricci curvature to a metric with constant sectional curvature. Many papers which are related to evolution equation (1) have been published since 1982. For examples one can see [9], [10], [22], [23], [24], [40], [41] and [42].

In [43] we proved that under the assumption of Theorem 1.1, the evolution equation (1) has a solution $g_{ij}(t)$ for all time $0 \le t < +\infty$, and the curvature of $g_{ij}(t)$ tends to zero as time $t \to +\infty$. We then constructed a flat Kähler metric on M. Thus we know that M is biholomorphic to $\mathbb{C}^n$.

After the graduation of the author from Harvard University in 1990, we continue to work to improve the result in Theorem 1.1. We have already found that under some weaker assumptions than that of Theorem 1.1, the evolution equation (1) still has a solution $g_{ij}(t)$ for all time $0 \le t < +\infty$. But to study the behavior of the solution $g_{ij}(t)$ as the time $t \to +\infty$ is a complicated problem. This problem is now partially solved by the use of the results of Andersen–Lempert [1] and Forstneric–Rosay [16] in 1992 and 1993. In their papers [1] and [16] some approximations of biholomorphic mappings by automorphisms of $\mathbb{C}^n$ were obtained. With the help of their results, we are going to prove the following main result in this paper:

Theorem 1.2. *Suppose M is a complete noncompact Kähler manifold of complex dimension n with bounded and positive sectional curvature. Suppose there exist constants $0 < \varepsilon,\ C_1 < +\infty$ such that*

$$\int_{B(x_0,\gamma)} R(x)dx \le \frac{C_1}{(\gamma+1)^{1+\varepsilon}} \cdot \mathrm{Vol}(B(x_0,\gamma)), \quad x_0 \in M,\ 0 \le \gamma < +\infty.$$

Then M is biholomorphic to a pseudoconvex domain in $\mathbb{C}^n$.

Because $\mathbb{C}^n$ is biholomorphic to some proper subdomains Ω of $\mathbb{C}^n$ when $n \ge 2$. These domains Ω are called Fatou–Bieberbach domains. For examples of Fatou–Bieberbach domains one can see Bochner–Martin [6], Dixon–Esterle [15] and Rosay–Rudin [38]. Thus to construct a biholomorphic map from the manifold M onto $\mathbb{C}^n$ is somehow difficult. If we can prove that the pseudoconvex domain which the manifold M is biholomorphic to in Theorem 1.2 is a Fatou–Bieberbach domain, then we know that the manifold M in Theorem 1.2 is actually biholomorphic to $\mathbb{C}^n$. This might be a topic for the future study.

In this paper, §2–§7 are modifications of the techniques appeared in [43] in 1990. Therefore, §2–§7 of this paper can be regarded as a

modified version of the author's thesis [43]. The result in Theorem 1.1 of this paper was announced in [44]. The author would like to thank Professor S.T. Yau for his suggestions and encouragement during my Ph.D. degree study program at Harvard University. The thanks are also due to Department of Mathematics, Harvard University and Alfred P. Sloan Foundation for their financial support during the proof of Theorem 1.1 in 1989 and 1990.

§9 of this paper contains an application of the results of Andersen–Lempert [1] and Forstneric–Rosay [16] in 1992 and 1993. With the help of their results on approximations of biholomorphic mappings by automorphisms of $\mathbb{C}^n$, we complete the proof of Theorem 1.2.

We talked about the result of this paper in the Workshop on Riemannian Metrics Satisfying Curvature Equations held at MSRI at Berkeley in September, 1993, and also in the Midwest Several Complex Variables Conference held at Purdue University in May, 1994.

2. Short time existence for the evolution equation

Suppose M is a Riemannian manifold with the metric

$$ds^2 = g_{ij}(x) dx^i dx^j > 0. \tag{1}$$

We use $\{R_{ijkl}\}$ to denote the Riemannian curvature tensor of M, and let

$$R_{ij} = g^{kl} R_{ikjl}, \quad R = g^{ij} R_{ij} = g^{ij} g^{kl} R_{ikjl}$$

be the Ricci curvature and scalar curvature respectively, where $(g^{ij}) = (g_{ij})^{-1}$.

For any two tensors such as $\{T_{ijkl}\}$ and $\{U_{ijkl}\}$ defined on M, we have the inner product

$$< T_{ijkl},\ U_{ijkl} >= g^{i\alpha} g^{j\beta} g^{k\gamma} g^{l\delta} T_{ijkl} U_{\alpha\beta\gamma\delta}.$$

The norm of $\{T_{ijkl}\}$ is defined as follows:

$$|T_{ijkl}|^2 =< T_{ijkl},\ T_{ijkl} > .$$

We use ∇T_{ijkl} to denote the covariant derivatives of the tensor $\{T_{ijkl}\}$ with respect to the metric ds^2, and $\nabla^m T_{ijkl}$ to denote all of the m–th order covariant derivatives of $\{T_{ijkl}\}$.

Using the evolution equation to deform the metric on any real n–dimensional Riemannian manifold (M, g_{ij}):

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{2}$$

the first important thing we have to consider is the short time existence for the solution of the evolution equation (2). In the case where M is a compact Riemannian manifold, Hamilton in [22] proved that for any given initial data metric g_{ij} on M, the evolution equation (2) always has a unique solution for a short time interval. In the case where M is a complete noncompact Riemannian manifold, the short time existence for the solution of evolution equation (2) is not true in general. It is easy to find a complete noncompact Riemannian manifold (M, g_{ij}) such that on which the evolution equation (2) does not have any solution for an arbitrarily small time interval. If we assume that the curvature tensor on M is bounded by some constant, then the short time existence theorem for the solution of evolution equation (2) was proved by the author in [40]. We have

Theorem 2.1. *Suppose $(M, g_{ij}(x))$ is an n–dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying*

$$|R_{ijkl}|^2 \leq k_0, \qquad \text{on } M, \tag{3}$$

where $0 < k_0 < +\infty$ is a constant. Then there exists a constant $T(n, k_0) > 0$ depending only on n and k_0 such that the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \\ g_{ij}(x,0) = g_{ij}(x), \ \forall x \in M \end{cases} \tag{4}$$

has a smooth solution $g_{ij}(x,t) > 0$ for a short time $0 \leq t \leq T(n, k_0)$, and satisfies the following estimates: For any integers $m \geq 0$, there exist constants $C(n, m, k_0) > 0$ depending only on n, m and k_0 such that

$$\sup_{x \in M} |\nabla^m R_{ijkl}(x,t)|^2 \leq C(n, m, k_0) \left(\frac{1}{t}\right)^m, \qquad 0 \leq t \leq T(n, k_0). \tag{5}$$

Proof. This is Theorem 1.1 in [40]. q.e.d.

More explicitly we have the following corollary:

Corollary 2.2. *Suppose $(M, g_{ij}(x))$ is an n–dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\{R_{ijkl}\}$ satisfying*

$$|R_{ijkl}|^2 \leq k_0, \qquad \text{on } M,$$

where $0 < k_0 < +\infty$ is a constant. Then there exists a constant $0 < \theta_0(n) < +\infty$ depending only on n such that the evolution equation (4) has a smooth solution $g_{ij}(x,t) > 0$ for a short time $0 \leq t \leq \theta_0(n)/\sqrt{k_0}$ and satisfies the following estimates: For any integers $m \geq 0$, there exist constants $C(n,m) > 0$ depending only on n and m such that

$$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \leq \frac{C(n,m)\cdot k_0}{t^m}, \qquad \text{for } 0 \leq t \leq \frac{\theta_0(n)}{\sqrt{k_0}}. \tag{6}$$

Proof. If $k_0 = 1$, Corollary 2.2 follows directly from Theorem 2.1. If $k_0 \neq 1$, we define a new metric on M:

$$\tilde{g}_{ij}(x) = \sqrt{k_0} g_{ij}(x), \qquad x \in M. \tag{7}$$

We use $\{\tilde{R}_{ijkl}(x)\}$ and $\tilde{\nabla}$ to denote, respectively, the Riemannian curvature tensor and the covariant derivative with respect to $\tilde{g}_{ij}(x)$. From the definition of $\tilde{g}_{ij}(x)$ it follows that

$$|\tilde{R}_{ijkl}(x)|^2 \leq 1, \qquad \text{on } M. \tag{8}$$

Using Theorem 2.1 we know that the evolution equation

$$\begin{cases} \frac{\partial}{\partial t}\tilde{g}_{ij}(x,t) = -2\tilde{R}_{ij}(x,t), \\ \tilde{g}_{ij}(x,0) = \tilde{g}_{ij}(x), \quad \forall x \in M \end{cases} \tag{9}$$

has a smooth solution $\tilde{g}_{ij}(x,t) > 0$ for a short time $0 \leq t \leq \theta_0(n)$, where $0 < \theta_0(n) < +\infty$ depends only on n. We still have

$$\sup_{x\in M} |\tilde{\nabla}^m \tilde{R}_{ijkl}(x,t)|^2 \leq \frac{C(n,m)}{t^m}, \quad 0 \leq t \leq \theta_0(n), \tag{10}$$

for all integers $m \geq 0$. Now we define

$$g_{ij}(x,t) = \frac{1}{\sqrt{k_0}}\tilde{g}_{ij}(x, \sqrt{k_0}t),\ x \in M,\ 0 \leq t \leq \frac{\theta_0(n)}{\sqrt{k_0}}. \tag{11}$$

Then it is easy to see that $g_{ij}(x,t) > 0$ is a smooth solution of the evolution equation (4) on $0 \leq t \leq \theta_0(n)/\sqrt{k_0}$ and satisfies (6) for any integers $m \geq 0$. q.e.d.

Lemma 2.3. *Suppose M is an n–dimensional complete noncompact Riemannian manifold, and $g_{ij}(x,t) > 0$ are smooth Riemannian metrics defined on $M \times [0,T]$, where $0 < T < +\infty$ is an arbitrary constant. Suppose the following assumptions hold:*

$$\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t), \qquad \text{on } M \times [0,T], \tag{12}$$

$$\sup_{M\times[0,T]} |R_{ijkl}(x,t)|^2 \leq k_0, \tag{13}$$

where $0 < k_0 < +\infty$ is a constant. Then for any integers $m \geq 1$, there exist constants $0 < c(n,m) < +\infty$ depending only on n and m such that

$$e^{-2n\sqrt{k_0}T}g_{ij}(x,0) \leq g_{ij}(x,t) \leq e^{2n\sqrt{k_0}T}g_{ij}(x,0), \quad x \in M,\ 0 \leq t \leq T, \tag{14}$$

$$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \leq c(n,m)\left[k_0 \cdot \left(\frac{1}{t}\right)^m + k_0^{\frac{m}{2}+1}\right], \quad 0 \leq t \leq T. \tag{15}$$

Proof. We can assume without lose of generality that $k_0 = 1$. If $k_0 \neq 1$, we can use the rescaling technique as what we did in the proof of Corollary 2.2. Thus we only need to prove Lemma 2.3 for the case $k_0 = 1$. From (13) it follows that

$$|R_{ijkl}(x,t)|^2 \leq 1, \quad \text{on } M \times [0,T]. \tag{16}$$

Thus

$$|R_{ij}(x,t)|^2 \leq n^2, \quad \text{on } M \times [0,T], \tag{17}$$

which, together with (12), yields

$$\left|\frac{\partial}{\partial t}g_{ij}(x,t)\right| \leq 2n, \qquad \text{on } M \times [0,T],$$

$$-2ng_{ij}(x,t) \leq \frac{\partial}{\partial t}g_{ij}(x,t) \leq 2ng_{ij}(x,t), \quad \text{on } M \times [0,T],$$

$$(18)\qquad e^{-2nt}g_{ij}(x,0)\le g_{ij}(x,t)\le e^{2nt}g_{ij}(x,0),\ \text{on } M\times[0,T].$$

Since $0\le t\le T$, from (18) it follows that

$$e^{-2nT}g_{ij}(x,0)\le g_{ij}(x,t)\le e^{2nT}g_{ij}(x,0),\quad \text{on } M\times[0,T].$$

Thus (14) is true for the case $k_0=1$. q.e.d.

Using (16), (18) and the same arguments as what we used in the proof of Lemma 7.1 in [40] we know that there exists a constant $0<\theta(n)<+\infty$ depending only on n such that for any integers $m\ge 1$, we have

$$(19)\qquad \sup_{x\in M}|\nabla^m R_{ijkl}(x,t)|^2\le \frac{c(n,m)}{t^m},\ 0\le t\le\theta(n),$$

where $0<c(n,m)<+\infty$ are constants depending only on n and m.

If $T\le\theta(n)$, then (15) is already true for the case $k_0=1$ by (19). If $T>\theta(n)$, for any $t_0\in[\theta(n),T]$, we define a new metric

$$(20)\ \tilde g_{ij}(x,t)=g_{ij}(x,t+t_0-\theta(n)),\ x\in M,\ \theta(n)-t_0\le t\le T+\theta(n)-t_0.$$

Combining (12), (16) and (20) gives

$$(21)\qquad \frac{\partial}{\partial t}\tilde g_{ij}(x,t)=-2\tilde R_{ij}(x,t),\ 0\le t\le T+\theta(n)-t_0,$$

$$(22)\qquad |\tilde R_{ijkl}(x,t)|^2\le 1,\quad \text{on } M\times[0,\ T-t_0+\theta(n)],$$

where we have used $\{\tilde R_{ijkl}(x,t)\}$ to denote the curvature tensor of $\tilde g_{ij}(x,t)$. Thus by the same reason as (19) we get

$$(23)\qquad \sup_{x\in M}|\tilde\nabla^m\tilde R_{ijkl}(x,t)|^2\le\frac{c(n,m)}{t^m},\ 0\le t\le\theta(n),\ m\ge 1.$$

Combining (20) and (23) yields

$$(24)\qquad \sup_{x\in M}|\nabla^m R_{ijkl}(x,t)|^2\le c(n,m)\left(\frac{1}{t-t_0+\theta(n)}\right)^m,$$

for all integers $m\ge 1$ and $t_0-\theta(n)\le t\le t_0$. Now we let $t=t_0$, from (24) it follows that

$$(25)\qquad \sup_{x\in M}|\nabla^m R_{ijkl}(x,t_0)|^2\le c(n,m)\left(\frac{1}{\theta(n)}\right)^m.$$

Since $t_0 \in [\theta(n), T]$ is arbitrary, by (25) for any integers $m \geq 1$, there exist constants $0 < \tilde{c}(n,m) < +\infty$ depending only on n and m such that

$$\sup_{x \in M} |\nabla^m R_{ijkl}(x,t)|^2 \leq \tilde{c}(n,m), \;\; \theta(n) \leq t \leq T. \tag{26}$$

Combining (19) and (26) we know that (15) is true for any T for the case $k_0 = 1$, and hence complete the proof of Lemma 2.3.

Now we start to discuss Kähler manifolds case. Suppose M is a complete Kähler manifold of complex dimension n with the Kähler metric

$$d\tilde{s}^2 = \tilde{g}_{\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta > 0, \tag{27}$$

where $z = \{z^1, z^2, \ldots, z^n\}$ denotes the local holomorphic coordinate on M. Suppose

$$\begin{cases} z^k = x^k + \sqrt{-1} x^{k+n}, \\ x^k \in \mathbb{R}, \; x^{k+n} \in \mathbb{R}, \end{cases} \qquad k = 1, 2, \ldots, n. \tag{28}$$

Then $x = \{x^1, x^2, \ldots, x^{2n}\}$ is the local real coordinate on M. Usually we use $\alpha, \beta, \gamma, \delta, \ldots$, etc. to denote the indices corresponding to holomorphic vectors or holomorphic covectors, $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}, \ldots$, etc. the indices corresponding to antiholomorphic vectors or antiholomorphic covectors, and i, j, k, $l, \ldots$, etc. the indices corresponding to real vectors or real covectors. Suppose in the real coordinate $x = \{x^i\}$ the Kähler metric (27) can be written as

$$d\tilde{s}^2 = \tilde{g}_{ij}(x) dx^i dx^j > 0. \tag{29}$$

Then (29) is a complete Riemannian metric on M, and M is a real $2n$–dimensional Riemannian manifold with this metric.

Applying to Kähler manifolds the result which we obtained for real Riemannian manifolds, we have

Theorem 2.4. *Suppose $(M, \tilde{g}_{\alpha\overline{\beta}}(x))$ is a complete noncompact Kähler manifold of complex dimension n with bounded and nonnegative holomorphic bisectional curvature:*

$$0 \leq -\tilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(x) \leq k_0, \quad \forall x \in M, \tag{30}$$

where $0 < k_0 < +\infty$ *is a constant. Then there exists a constant* $0 < \theta_0(n) < +\infty$ *depending only on* n *such that the evolution equation*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \\ g_{ij}(x,0) = \tilde{g}_{ij}(x), \quad \forall x \in M \end{cases} \tag{31}$$

has a smooth solution $g_{ij}(x,t) > 0$ *for a short time*

$$0 \le t \le \frac{\theta_0(n)}{k_0}, \tag{32}$$

and satisfies the following estimates: For any integers $m \ge 0$*, there exist constants* $c(n,m) > 0$ *depending only on* n *and* m *such that*

$$\sup_{x \in M} |\nabla^m R_{ijkl}(x,t)|^2 \le \frac{c(n,m) \cdot k_0^2}{t^m}, \; 0 \le t \le \frac{\theta_0(n)}{k_0}. \tag{33}$$

Proof. Since $-\tilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(x) \ge 0$ on M, using (30) it is easy to see that

$$|\tilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}(x)|^2 \le 400 n^4 k_0^2, \quad \forall x \in M. \tag{34}$$

If we write it in the real coordinate, we have

$$|\tilde{R}_{ijkl}(x)|^2 \le 40000 n^4 k_0^2, \quad \forall x \in M. \tag{35}$$

Thus from Corollary 2.2 and (35) it follows that Theorem 2.4 is true.
q.e.d.

3. The construction of exhaustion functions

In the previous section, we established the short time existence theorem for the solution of Ricci flow on complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature. To control the solution and prove the long time existence for the solution of Ricci flow, we need to construct some good smooth exhaustion functions on the manifold. For that purpose we use the results and the techniques which were derived by R. Schoen and S.T. Yau in their book [39], and also the iteration arguments of J. Moser [35].

Suppose $(M,\ g_{ij}(x))$ is an n–dimensional complete Riemannian manifold. We use ∇ to denote the covariant derivatives with respect to the metric g_{ij}, and

$$\Delta = g^{ij} \nabla_i \nabla_j \tag{1}$$

the Laplacian operator with respect to the metric g_{ij} on M. For any two points x_0, $x \in M$, let $\gamma(x, x_0)$ denote the distance between x_0 and x. For any point $x \in M$ and $\gamma > 0$, let $B(x, \gamma)$ denote the geodesic ball of radius γ and centered at x:

$$B(x,\gamma) = \{y \in M | \gamma(x,y) < \gamma\}. \tag{2}$$

Now we have the result of Schoen–Yau [39]:

Theorem 3.1. *Suppose* $(M,\ g_{ij}(x))$ *is an* n*–dimensional complete noncompact Riemannian manifold with its Ricci curvature bounded from below:*

$$R_{ij}(x) \geq -k_0 g_{ij}(x), \quad \forall x \in M, \tag{3}$$

where $0 \leq k_0 < +\infty$ *is a constant. Then there exists a constant* $0 < C_2 < +\infty$ *depending only on* n *and* k_0 *such that for any fixed point* $x_0 \in M$*, there exists a smooth function* $\varphi(x) \in C^\infty(M)$ *such that*

$$\begin{cases} \frac{1}{C_2}[1+\gamma(x,x_0)] \leq \varphi(x) \leq C_2[1+\gamma(x,x_0)], \\ |\nabla\varphi(x)| \leq C_2, \qquad \forall x \in M. \\ |\Delta\varphi(x)| \leq C_2, \end{cases} \tag{4}$$

Proof. This is Theorem 1.4.2 in the book of R. Schoen and S.T. Yau [39]. Since that book [39] is in Chinese, we sketch their proof here.

Suppose $\lambda > 0$ is a constant to be determined later and $\gamma > 1$. We try to solve the following Dirichlet problem:

$$\begin{cases} \Delta\mathcal{U}_\gamma(x) = \lambda\mathcal{U}_\gamma(x), & x \in B(x_0,\gamma)\backslash B(x_0,1), \\ \mathcal{U}_\gamma(x) \equiv 0, & x \in \partial B(x_0,\gamma), \\ \mathcal{U}_\gamma(x) \equiv 1, & x \in \partial B(x_0,1), \end{cases} \tag{5}$$

where $\partial B(x_0,\gamma)$ denotes the boundary of $B(x_0,\gamma)$. If $\partial B(x_0,1)$ or $\partial B(x_0,\gamma)$ has some singular points, we just make a small perturbation of them such that the boundaries become smooth. Using the classical theory of the second order elliptic equations we know that (5) has a smooth solution $\mathcal{U}_\gamma(x)$ on $B(x_0,\gamma)\backslash B(x_0,1)$. By the maximum principle we have

$$0 < \mathcal{U}_\gamma(x) < 1, \quad x \in B(x_0,\gamma)\backslash\overline{B(x_0,1)}. \tag{6}$$

If $\gamma_2 > \gamma_1 > 1$, then

(7)
$$\begin{cases} \Delta[\mathcal{U}_{\gamma_2}(x) - \mathcal{U}_{\gamma_1}(x)] = \lambda[\mathcal{U}_{\gamma_2}(x) - \mathcal{U}_{\gamma_1}(x)], & \text{for } x \in B(x_0, \gamma_1)\backslash B(x_0, 1), \\ \mathcal{U}_{\gamma_2}(x) - \mathcal{U}_{\gamma_1}(x) \equiv 0, & x \in \partial B(x_0, 1), \\ \mathcal{U}_{\gamma_2}(x) - \mathcal{U}_{\gamma_1}(x) = \mathcal{U}_{\gamma_2}(x) > 0, & x \in \partial B(x_0, \gamma_1). \end{cases}$$

Using the maximum principle again yields

$$\mathcal{U}_{\gamma_2}(x) - \mathcal{U}_{\gamma_1}(x) > 0, \quad \text{for } x \in B(x_0, \gamma_1)\backslash\overline{B(x_0, 1)}. \tag{8}$$

Combining (6) and (8) shows that as $\gamma \to +\infty$ the limit

$$\mathcal{U}(x) = \lim_{\gamma \to +\infty} \mathcal{U}_\gamma(x) \tag{9}$$

exists for any $x \in M\backslash B(x_0, 1)$, and satisfies

$$0 < \mathcal{U}(x) \le 1, \quad \forall x \in M\backslash B(x_0, 1). \tag{10}$$

For any point $x_1 \in M$ and $\delta > 0$, $\gamma > 1$, if the following condition holds:

$$B(x_1, \delta) \subset B(x_0, \gamma)\backslash\overline{B(x_0, 1)}, \tag{11}$$

then from (5) and (6) we have

$$\begin{cases} \Delta\mathcal{U}_\gamma(x) = \lambda\mathcal{U}_\gamma(x), & x \in B(x_1, \delta), \\ 0 < \mathcal{U}_\gamma(x) < 1, & x \in B(x_1, \delta). \end{cases} \tag{12}$$

By Theorem 6 in [12] for the gradient estimates of the solutions of elliptic equations,

$$|\nabla\mathcal{U}_\gamma(x)| \le C(n, \delta, k_0, \lambda) \cdot \mathcal{U}_\gamma(x), \quad \forall x \in B(x_1, \frac{\delta}{2}), \tag{13}$$

where $0 < C(n, \delta, k_0, \lambda) < +\infty$ is a constant depending only on n, δ, k_0 and λ. (13) can be written as

$$\sup_{x \in B(x_1, \frac{\delta}{2})} |\nabla \log \mathcal{U}_\gamma(x)| \le C(n, \delta, k_0, \lambda). \tag{14}$$

Since $0 < \mathcal{U}_\gamma(x) < 1$, from (14) it follows that

$$\sup_{x \in B(x_1, \frac{\delta}{2})} |\nabla\mathcal{U}_\gamma(x)| \le C(n, \delta, k_0, \lambda). \tag{15}$$

Combining (12), (15) and the classical Schauder estimates for the solutions of elliptic equations yields that for any integers $m \geq 2$, there exist constants $0 < C(n, m, \delta, g_{ij}|_{B(x_1,\delta)}) < +\infty$ depending only on n, m, δ and the metric g_{ij} on $B(x_1, \delta)$ such that

$$\sup_{x \in B(x_1, \frac{1}{4}\delta + (\frac{1}{2})^{m+1} \cdot \delta)} |\nabla^m \mathcal{U}_\gamma(x)| \leq C(n, m, \delta, g_{ij}|_{B(x_1,\delta)}). \tag{16}$$

which, together with (15), implies that all of the covariant derivatives of $\mathcal{U}_\gamma(x)$ are uniformally bounded on any compact subsets of $M \backslash \overline{B(x_0, 1)}$ as $\gamma \to +\infty$. Thus by Ascoli–Arzela's lemma, there exists a subsequence $\{\gamma_i\}$, $\gamma_i \to +\infty$ as $i \to +\infty$ such that

$$\mathcal{U}_{\gamma_i}(x) \overset{C^\infty}{\to} \longrightarrow \mathcal{U}(x), \quad \text{on } M \backslash \overline{B(x_0, 1)}, \quad \text{as } i \to +\infty, \tag{17}$$

where $\mathcal{U}(x)$ is defined by (9). Combining (5) and (17) we obtain

$$\begin{cases} \mathcal{U}(x) \in C^\infty(M \backslash \overline{B(x_0, 1)}), \\ \Delta \mathcal{U}(x) = \lambda \mathcal{U}(x), \quad x \in M \backslash \overline{B(x_0, 1)}. \end{cases} \tag{18}$$

From the classical theory of elliptic equations it follows that $\mathcal{U}_\gamma(x)$ in (5) are continuous up to the boundary $\partial B(x_0, 1)$. Thus combining (8), (9), (10) and (18) yields

$$\begin{cases} \mathcal{U}(x) \in C^0(M \backslash B(x_0, 1)), \\ \mathcal{U}(x) \equiv 1, \quad x \in \partial B(x_0, 1), \\ 0 < \mathcal{U}(x) < 1, \quad x \in M \backslash \overline{B(x_0, 1)}. \end{cases} \tag{19}$$

From (14) we still have

$$\sup_{x \in B(x_1, \frac{\delta}{2})} |\nabla \log \mathcal{U}(x)| \leq C(n, \delta, k_0, \lambda). \tag{20}$$

Since $x_1 \in M \backslash B(x_0, 1 + \delta)$ is arbitrary, we get

$$\sup_{x \in M \backslash B(x_0, 1+\delta)} |\nabla \log \mathcal{U}(x)| \leq C(n, \delta, k_0, \lambda), \quad \forall \delta > 0. \tag{21}$$

Now we are going to show that $\mathcal{U}(x)$ actually tends to zero exponentially as $x \to \infty$ if λ is large enough.

Lemma 3.2. *Suppose M is an n–dimensional complete noncompact Riemannian manifold with its Ricci curvature $R_{ij}(x)$ satisfying*

$$R_{ij}(x) \geq -k_0 g_{ij}(x), \quad \forall x \in M, \tag{22}$$

where $0 \leq k_0 < +\infty$ *is a constant. Then there exists a constant* $0 < C_4 < +\infty$ *depending only on* n *and* k_0 *such that*

$$\operatorname{Vol} B(x,1) \geq e^{-C_4\gamma(x,x_0)} \cdot \operatorname{Vol} B(x_0,1) \tag{23}$$

for any $x, x_0 \in M$, *where Vol* $B(x,1)$ *denotes the volume of the geodesic ball* $B(x,1)$.

Proof. For a fixed point $x \in M$ and any $y \in M$ we define a function

$$\rho(y) = \gamma(x,y). \tag{24}$$

Since $R_{ij} \geq -k_0$ on M, using the Laplacian operator comparison theorem we obtain

$$\begin{cases} \Delta\rho(y) \leq \frac{n-1}{\rho(y)} + \sqrt{(n-1)k_0}, \\ \\ |\nabla\rho(y)| \leq 1, \end{cases} \qquad \forall y \in M. \tag{25}$$

At the nonsmooth points of $\rho(y)$, we know that (25) is still true in the sense of distribution. Thus

$$\Delta\rho^2 \leq 2n + 2\sqrt{(n-1)k_0}\rho, \qquad \text{on } M, \tag{26}$$

and for any $t > 0$,

$$\int_{B(x,t)} \Delta\rho(y)^2 dy \leq \int_{B(x,t)} 2n dy + 2\sqrt{(n-1)k_0} \int_{B(x,t)} \rho(y) dy, \tag{27}$$

$$\int_{B(x,t)} \Delta\rho(y)^2 dy \leq 2n \cdot \operatorname{Vol} B(x,t) + 2t\sqrt{(n-1)k_0} \cdot \operatorname{Vol} B(x,t). \tag{28}$$

By the Stokes theorem we have

$$\int_{B(x,t)} \Delta\rho(y)^2 dy = \int_{\partial B(x,t)} \frac{\partial\rho^2}{\partial t} = 2t \cdot \operatorname{Vol}(\partial B(x,t)). \tag{29}$$

On the other hand,

$$\operatorname{Vol}(\partial B(x,t)) = \frac{\partial}{\partial t} \operatorname{Vol} B(x,t). \tag{30}$$

Combining (28), (29) and (30) gives

$$(31)\qquad 2t\frac{\partial}{\partial t}\operatorname{Vol} B(x,t) \le 2n\cdot \operatorname{Vol} B(x,t) + 2t\sqrt{(n-1)k_0}\cdot \operatorname{Vol} B(x,t),$$

$$(32)\qquad t\frac{\partial}{\partial t}\operatorname{Vol} B(x,t) \le n\cdot \operatorname{Vol} B(x,t) + t\sqrt{(n-1)k_0}\cdot \operatorname{Vol} B(x,t),$$

$$(33)\qquad \frac{\partial}{\partial t}[t^{-n}\cdot e^{-\sqrt{(n-1)k_0}\cdot t}\cdot \operatorname{Vol} B(x,t)] \le 0,\ \ 0\le t<+\infty.$$

Thus if $t\ge 1$, then

$$(34)\qquad t^{-n}e^{-\sqrt{(n-1)k_0}\cdot t}\cdot \operatorname{Vol} B(x,t) \le e^{-\sqrt{(n-1)k_0}}\cdot \operatorname{Vol} B(x,1).$$

Now we choose $t=\gamma(x,x_0)+1$, by (34) we obtain

$$(35)\qquad \begin{aligned}&[1+\gamma(x,x_0)]^{-n}\cdot e^{-\sqrt{(n-1)k_0}[1+\gamma(x,x_0)]}\cdot \operatorname{Vol} B(x,1+\gamma(x,x_0))\\ &\qquad \le e^{-\sqrt{(n-1)k_0}}\cdot \operatorname{Vol} B(x,1).\end{aligned}$$

Since $B(x_0,1)\subset B(x,1+\gamma(x,x_0))$, from (35) it follows that

$$(36)\qquad \operatorname{Vol} B(x,1) \ge [1+\gamma(x,x_0)]^{-n}\cdot e^{-\sqrt{(n-1)k_0}\cdot\gamma(x,x_0)}\cdot \operatorname{Vol} B(x_0,1).$$

Thus (23) is true. q.e.d.

Coming back to the proof of Theorem 3.1, from Lemma 3.2 we know that there exists a constant $0<C_4<+\infty$ depending only on n and k_0 such that

$$(37)\qquad \operatorname{Vol} B(x,1) \ge e^{-C_4\gamma(x,x_0)}\cdot \operatorname{Vol} B(x_0,1),\quad \forall x, x_0\in M.$$

Suppose $0<a<+\infty$ is a constant to be determined later, and $\mathcal{U}_\gamma(x)$ are the solutions of (5) for $\gamma\ge 3$. Using the Stokes theorem and (5) we have

$$\begin{aligned}\int_{B(x_0,\gamma)\backslash B(x_0,2)} & e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)\Delta\mathcal{U}_\gamma(x)\cdot dx\\ =&\int_{\partial B(x_0,2)} e^{a\gamma(x,x_0)}\cdot\mathcal{U}_\gamma(x)\cdot\frac{\partial\mathcal{U}_\gamma(x)}{\partial\nu}\\ &-\int_{B(x_0,\gamma)\backslash B(x_0,2)}\nabla_i[e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)]\cdot\nabla_i\mathcal{U}_\gamma(x)dx\\ =&e^{2a}\int_{\partial B(x_0,2)}\mathcal{U}_\gamma(x)\cdot\frac{\partial\mathcal{U}_\gamma(x)}{\partial\nu}\\ &-\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}|\nabla\mathcal{U}_\gamma(x)|^2dx\\ &-a\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)\cdot\nabla_i\gamma(x,x_0)\cdot\nabla_i\mathcal{U}_\gamma(x)\cdot dx,\end{aligned}\tag{38}$$

where ν is the outer unit normal vector of $\partial B(x_0,2)$. Thus

$$\left|\frac{\partial\mathcal{U}_\gamma(x)}{\partial\nu}\right|\leq|\nabla\mathcal{U}_\gamma(x)|,\quad\forall\ x\in\partial B(x_0,2).\tag{39}$$

From (15) it follows that

$$\sup_{x\in B(x_0,\gamma-\frac{1}{2})\backslash B(x_0,2)}|\nabla\mathcal{U}_\gamma(x)|\leq C(n,k_0,\lambda).\tag{40}$$

Combining (39) and (40) yields

$$\sup_{x\in\partial B(x_0,2)}\left|\frac{\partial\mathcal{U}_\gamma(x)}{\partial\nu}\right|\leq C(n,k_0,\lambda).\tag{41}$$

Since $0\leq\mathcal{U}_\gamma(x)\leq 1$, by (41) we get

$$e^{2a}\int_{\partial B(x_0,2)}\mathcal{U}_\gamma(x)\frac{\partial\mathcal{U}_\gamma(x)}{\partial\nu}\leq C(n,k_0,\lambda)\cdot e^{2a}\cdot\mathrm{Vol}(\partial B(x_0,2)).\tag{42}$$

Since $\Delta\mathcal{U}_\gamma(x)=\lambda\mathcal{U}_\gamma(x)$ on $B(x_0,\gamma)\backslash B(x_0,2)$, from (38) and (42) we know that

$$\begin{aligned}
&\lambda\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)^2dx\\
&=\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)\Delta\mathcal{U}_\gamma(x)dx\\
&\le C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2))\\
&\quad -\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}|\nabla\mathcal{U}_\gamma(x)|^2dx\\
&\quad -\int_{B(x_0,\gamma)\backslash B(x_0,2)} ae^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)\cdot\nabla_i\gamma(x,x_0)\cdot\nabla_i\mathcal{U}_\gamma(x)dx,\\
&\qquad\qquad\qquad\qquad \text{for } \gamma\ge 3.
\end{aligned}\tag{43}$$

Since we still have

$$|\nabla\gamma(x,x_0)|\le 1,\quad \forall\ x\in M,\tag{44}$$

combining (43) and (44) gives

$$\begin{aligned}
&\lambda\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)^2dx\\
&\le C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2))\\
&\quad -\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}|\nabla\mathcal{U}_\gamma(x)|^2dx\\
&\quad +a\int_{B(x_0,\gamma)\backslash B(x_0,2)} \mathcal{U}_\gamma(x)\cdot e^{a\gamma(x,x_0)}|\nabla\mathcal{U}_\gamma(x)|dx\\
&\le C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2))\\
&\quad +\frac{a^2}{4}\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)^2dx.\\
&\quad \left(\lambda-\frac{a^2}{4}\right)\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)^2dx\\
&\le C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2)),\quad \text{for any } \gamma\ge 3.
\end{aligned}\tag{45}$$

Now we choose

$$\lambda=\frac{a^2}{4}+1.\tag{46}$$

Then by (45) we get

$$\int_{B(x_0,\gamma)\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}_\gamma(x)^2dx$$

$$\leq C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2)), \quad \text{for } \gamma\geq 3. \tag{47}$$

Let $\gamma\to+\infty$. Then from (17) and (47) it follows that

$$\int_{M\backslash B(x_0,2)} e^{a\gamma(x,x_0)}\mathcal{U}(x)^2dx \leq C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2)). \tag{48}$$

For any point $y\in M\backslash B(x_0,3)$, we want to use (48) to estimate $\mathcal{U}(y)$. Since $y\in M\backslash B(x_0,3)$, we have

$$B(y,1)\subset M\backslash B(x_0,2), \tag{49}$$

$$\gamma(x,x_0)\geq\gamma(y,x_0)-1, \quad \forall\ x\in B(y,1). \tag{50}$$

Combining (48) and (49) yields

$$\int_{B(y,1)} e^{a\gamma(x,x_0)}\mathcal{U}(x)^2dx \leq C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2)). \tag{51}$$

From (50) and (51) it follows that

$$e^{a[\gamma(y,x_0)-1]}\int_{B(y,1)}\mathcal{U}(x)^2dx \leq C(n,k_0,\lambda)\cdot e^{2a}\cdot \mathrm{Vol}(\partial B(x_0,2)),$$

(52)

$$\int_{B(y,1)}\mathcal{U}(x)^2dx \leq C(n,k_0,\lambda)\cdot e^{3a-a\gamma(y,x_0)}\cdot \mathrm{Vol}(\partial B(x_0,2)).$$

Using gradient estimate (21) we get

$$|\log\mathcal{U}(x)-\log\mathcal{U}(y)|\leq C(n,k_0,\lambda), \quad \forall\ x\in B(y,1). \tag{53}$$

$$\mathcal{U}(x)\geq e^{-C(n,k_0,\lambda)}\mathcal{U}(y), \quad \forall\ x\in B(y,1). \tag{54}$$

Combining (52) and (54) gives

$$e^{-2C(n,k_0,\lambda)}\cdot\mathcal{U}(y)^2\cdot \mathrm{Vol}\,B(y,1)$$

$$\leq C(n,k_0,\lambda)\cdot e^{3a-a\gamma(y,x_0)}\cdot \mathrm{Vol}(\partial B(x_0,2)).$$

(55)

$$\mathcal{U}(y)\leq C(n,k_0,\lambda)^{\frac{1}{2}}\cdot e^{\frac{3a}{2}+C(n,k_0,\lambda)}\cdot e^{-\frac{a}{2}\gamma(y,x_0)}$$

$$\cdot\left[\frac{\mathrm{Vol}(\partial B(x_0,2))}{\mathrm{Vol}\,B(y,1)}\right]^{\frac{1}{2}}, \quad \forall\ y\in M\backslash B(x_0,3).$$

By (37) we obtain

$$\text{Vol}\, B(y,1) \geq e^{-C_4\gamma(y,x_0)} \cdot \text{Vol}\, B(x_0,1),\ y \in M,$$

where $0 < C_4 < +\infty$ depends only on n and k_0. Thus

$$\frac{\text{Vol}(\partial B(x_0,2))}{\text{Vol}\, B(y,1)} \leq e^{C_4\gamma(y,x_0)} \cdot \frac{\text{Vol}(\partial B(x_0,2))}{\text{Vol}\, B(x_0,1)},\ y \in M. \tag{56}$$

If we let $x = x_0$ and $t = 2$, from (28), (29) and (34) it follows respectively that

$$\begin{aligned} 4\,\text{Vol}(\partial B(x_0,2)) \leq & 2n \cdot \text{Vol}\, B(x_0,2) \\ & + 4\sqrt{(n-1)k_0} \cdot \text{Vol}\, B(x_0,2), \end{aligned} \tag{57}$$

$$\frac{\text{Vol}(\partial B(x_0,2))}{\text{Vol}\, B(x_0,2)} \leq \frac{n}{2} + \sqrt{(n-1)k_0}, \tag{58}$$

and

$$\begin{aligned} \left(\frac{1}{2}\right)^n e^{-2\sqrt{(n-1)k_0}} \cdot \text{Vol}\, B(x_0,2) & \\ \leq & e^{-\sqrt{(n-1)k_0}} \cdot \text{Vol}\, B(x_0,1), \end{aligned}$$

$$\frac{\text{Vol}\, B(x_0,2)}{\text{Vol}\, B(x_0,1)} \leq 2^n \cdot e^{\sqrt{(n-1)k_0}}. \tag{59}$$

Combining (58) and (59) yields

$$\frac{\text{Vol}(\partial B(x_0,2))}{\text{Vol}\, B(x_0,1)} \leq C_5(n,k_0), \tag{60}$$

where $0 < C_5(n,k_0) < +\infty$ depends only on n and k_0. By (55), (56) and (60) we have

$$\mathcal{U}(y) \leq C_6(n,k_0,\lambda,a) \cdot e^{-\frac{a}{2}\gamma(y,x_0)} \cdot e^{\frac{1}{2}C_4\gamma(y,x_0)}, \tag{61}$$

where $0 < C_6(n,k_0,\lambda,a) < +\infty$ depends only on n, k_0, λ and a. Now we choose

$$a = 2 + C_4. \tag{62}$$

Then a depends only on n and k_0. From (46) we know that

$$\lambda = 1 + \frac{1}{4}(2 + C_4)^2 \tag{63}$$

depends only on n and k_0. Combining (61), (62) and (63) we get

$$\mathcal{U}(y) \leq C_7(n, k_0) \cdot e^{-\gamma(y,x_0)}, \quad \forall\ y \in M \backslash B(x_0, 3), \tag{64}$$

where $0 < C_7(n, k_0) < +\infty$ depends only on n and k_0.

In (64) we obtain the upper bound estimate for $\mathcal{U}(x)$. Now we want to control $\mathcal{U}(x)$ from below.

Suppose $0 < m < +\infty$ is an integer to be determined later. $\gamma(x, x_0)$ denotes the distance between x and x_0. We define a function

$$f(x) = 1 - \frac{1}{\gamma(x, x_0)^m}, \quad x \in M \backslash \{x_0\}. \tag{65}$$

Then

$$\nabla_i f(x) = \frac{m}{\gamma(x, x_0)^{m+1}} \nabla_i \gamma(x, x_0). \tag{66}$$

$$\begin{aligned} \Delta f(x) &= \frac{m}{\gamma(x, x_0)^{m+1}} \Delta\gamma(x, x_0) - \frac{m(m+1)}{\gamma(x, x_0)^{m+2}} |\nabla_i \gamma(x, x_0)|^2 \\ &= \frac{m}{\gamma(x, x_0)^{m+1}} \left[\Delta\gamma(x, x_0) - \frac{(m+1)}{\gamma(x, x_0)} |\nabla\gamma(x, x_0)|^2 \right]. \end{aligned} \tag{67}$$

Since $R_{ij} \geq -k_0$ on M, using Laplacian operator comparison theorem we obtain

$$\begin{cases} \Delta\gamma(x, x_0) \leq \frac{n-1}{\gamma(x,x_0)} + \sqrt{(n-1)k_0}, \\ \\ |\nabla\gamma(x, x_0)| = 1, \ \text{a.e.} \end{cases} \qquad x \in M. \tag{68}$$

Combining (67) and (68) gives

$$\Delta f(x) \leq \frac{m}{\gamma(x, x_0)^{m+1}} \left[\frac{n-1}{\gamma(x, x_0)} + \sqrt{(n-1)k_0} - \frac{(m+1)}{\gamma(x, x_0)} \right], \quad x \in M \backslash \{x_0\}. \tag{69}$$

Now we choose an integer m such that

$$2\sqrt{(n-1)k_0} + n + 2(\lambda + 1) \leq m < 2\sqrt{(n-1)k_0} + n + 2(\lambda + 1) + 1, \tag{70}$$

where λ is defined by (63). Then for any points $x \in B(x_0, 2^{\frac{1}{m+1}})\backslash B(x_0, 1)$, we have

$$1 \leq \gamma(x, x_0) \leq 2^{\frac{1}{m+1}} \leq 2, \tag{71}$$

$$\begin{aligned} \frac{n-1}{\gamma(x,x_0)} + \sqrt{(n-1)k_0} - \frac{(m+1)}{\gamma(x,x_0)} \\ \leq \sqrt{(n-1)k_0} - \frac{(m-n+2)}{2} \leq -1. \end{aligned} \tag{72}$$

Combining (69), (70), (71) and (72) we get

$$\begin{aligned} \Delta f(x) \leq \frac{m}{\gamma(x,x_0)^{m+1}}(-1) \leq -\frac{m}{2} \leq -\lambda - 1, \\ \forall\ x \in B(x_0, 2^{\frac{1}{m+1}})\backslash B(x_0, 1). \end{aligned} \tag{73}$$

Remark. The function $f(x)$ defined in (65) may not be smooth at some points of $M\backslash\{x_0\}$. For example, if x is within the cut–locus of x_0, then $f(x)$ may not be smooth at x. But if we study the behavior of the distance function $\gamma(x, x_0)$ carefully, we know that at the nonsmooth points of $f(x)$, (69) and (73) are still true in the sense of distribution. Thus by making a small perturbation of $f(x)$ (for example, making a small perturbation of $f(x)$ by the use of mollifier technique) we can assume without loss of generality that $f(x)$ is a smooth function on $M\backslash\{x_0\}$, and (69) and (73) are true in the classical sense.

Since $0 < \mathcal{U}(x) \leq 1$ on $M\backslash B(x_0, 1)$, (18) implies

$$\Delta \mathcal{U}(x) \leq \lambda, \quad \forall\ x \in M\backslash\overline{B(x_0, 1)}. \tag{74}$$

Combining (73) and (74) yields

$$\Delta[\mathcal{U}(x) + f(x)] \leq -1, \quad \forall\ x \in B(x_0, 2^{\frac{1}{m+1}})\backslash\overline{B(x_0, 1)}. \tag{75}$$

By (19) and (65) we obtain

$$\mathcal{U}(x) + f(x) \equiv 1, \quad x \in \partial B(x_0, 1). \tag{76}$$

By (19), (65) and (71) we get

$$\begin{aligned} \mathcal{U}(x) + f(x) \geq f(x) = 1 - \left(\frac{1}{2}\right)^{\frac{m}{m+1}} \geq 1 - \frac{1}{\sqrt{2}} \geq \frac{1}{4}, \\ x \in \partial B(x_0, 2^{\frac{1}{m+1}}). \end{aligned} \tag{77}$$

Using the maximum principle, from (75), (76) and (77) we know that

$$\mathcal{U}(x) + f(x) \geq \frac{1}{4}, \ \forall \ x \in B(x_0, 2^{\frac{1}{m+1}}) \backslash B(x_0, 1). \tag{78}$$

For any $x \in B(x_0, (\frac{8}{7})^{\frac{1}{m}}) \backslash B(x_0, 1)$, by (65) we get $f(x) \leq \frac{1}{8}$. Thus (78) implies

$$\mathcal{U}(x) \geq \frac{1}{8}, \quad \forall \ x \in B(x_0, (\frac{8}{7})^{\frac{1}{m}}) \backslash B(x_0, 1). \tag{79}$$

On the other hand, from (21) it follows that

$$|\nabla \log \mathcal{U}(x)| \leq C_8(n, k_0, \lambda, m), \ \forall \ x \in M \backslash B(x_0, (\frac{8}{7})^{\frac{1}{2m}}). \tag{80}$$

Combining (79) and (80) gives

$$\mathcal{U}(x) \geq \frac{1}{8} e^{-C_8(n,k_0,\lambda,m) \cdot \gamma(x,x_0)}, \ x \in M \backslash B(x_0, 1), \tag{81}$$

where $0 < C_8(n, k_0, \lambda, m) < +\infty$ depends only on n, k_0, λ and m. From (63) and (70) we know that λ and m depend only on n and k_0. Thus (81) implies

$$\mathcal{U}(x) \geq e^{-C_9(n,k_0)\gamma(x,x_0)}, \ x \in M \backslash B(x_0, 1), \tag{82}$$

where $0 < C_9(n, k_0) < +\infty$ depends only on n and k_0.

Lemma 3.3. *Under the curvature assumption of Theorem 3.1, for any point $x_0 \in M$, there exists a smooth function $\mathcal{U}(x) \in C^\infty(M \backslash B(x_0, 2))$ such that*

$$\begin{cases} 0 < \mathcal{U}(x) < 1, \\ \Delta \mathcal{U}(x) = \lambda \mathcal{U}(x), \\ & \forall \ x \in M \backslash B(x_0, 3), \\ |\nabla \log \mathcal{U}(x)| \leq C_{10}(n, k_0), \\ \mathcal{U}(x) \leq C_{10}(n, k_0) \cdot e^{-\gamma(x,x_0)}, \\ \mathcal{U}(x) \geq e^{-C_{10}(n,k_0) \cdot \gamma(x,x_0)}, \end{cases} \tag{83}$$

where $0 < C_{10}(n, k_0) < +\infty$ depends only on n and k_0, λ is defined by (63).

Proof. Combining (18), (19), (21), (64) and (82) shows that the Lemma is true. q.e.d.

Under the curvature assumption of Theorem 3.1, for any fixed point $x_0 \in M$, suppose $\mathcal{U}(x) \in C^\infty(M \backslash B(x_0, 2))$ is the function obtained in Lemma 3.3. We then define another function $w(x) \in C^\infty(M \backslash B(x_0, 2))$ such that

$$w(x) = -\log \mathcal{U}(x) + \log C_{10}(n, k_0) + 1, \; x \in M \backslash B(x_0, 2). \tag{84}$$

By the definition of $w(x)$ we have

$$\nabla w(x) = -\nabla \log \mathcal{U}(x), \tag{85}$$

$$\Delta w(x) = -\Delta \log \mathcal{U}(x) = -\frac{\Delta \mathcal{U}(x)}{\mathcal{U}(x)} + |\nabla \log \mathcal{U}(x)|^2,$$

$$\Delta w(x) = -\lambda + |\nabla \log \mathcal{U}(x)|^2, \; x \in M \backslash B(x_0, 3). \tag{86}$$

Combining (83), (84), (85) and (86) we know that there exists a constant $0 < C_{11}(n, k_0) < +\infty$ depending only on n and k_0 such that

$$\begin{cases} 1 + \gamma(x, x_0) \le w(x) \le C_{11}[1 + \gamma(x, x_0)], \\ |\nabla w(x)| \le C_{11}, & \forall \; x \in M \backslash B(x_0, 3). \\ |\Delta w(x)| \le C_{11}, \end{cases} \tag{87}$$

To prove Theorem 3.1 the only thing we need to do is to try to extend the function $w(x)$ which we obtained in (87) to the whole manifold M in a suitable way such that we can still control $|\nabla w|$ and $|\Delta w|$ on the whole manifold M and only in terms of n and k_0. Suppose $y \in M$ is a point such that

$$\gamma(x_0, y) = 5(1 + C_{11}). \tag{88}$$

Using Lemma 3.3 again we can find another function $q(x) \in C^\infty(M \backslash B(y, 2))$ such that

$$\begin{cases} 1 + \gamma(x, y) \le q(x) \le C_{11}[1 + \gamma(x, y)], \\ |\nabla q(x)| \le C_{11}, & \forall \; x \in M \backslash B(y, 3). \\ |\Delta q(x)| \le C_{11}, \end{cases} \tag{89}$$

It is easy to find a smooth function $\theta(t) \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \theta(t) \equiv 0, & -\infty < t \le 5C_{11}, \\ 0 \le \theta(t) \le 1, & 5C_{11} \le t \le 2 + 5C_{11}, \\ \theta(t) \equiv 1, & 2 + 5C_{11} \le t < +\infty, \end{cases} \tag{90}$$

$$\begin{cases} |\theta'(t)| \le 1, & -\infty < t < +\infty, \\ |\theta''(t)| \le 4, & -\infty < t < +\infty. \end{cases} \tag{91}$$

Now we just define

$$
\text{(92)} \qquad \begin{cases} \varphi(x) = q(x), & \text{for } x \in B(x_0, \frac{7}{2}), \\ \varphi(x) = \theta(w(x)) \cdot w(x) + [1 - \theta(w(x))] \cdot q(x), & \\ & \text{for } x \in B(x_0, \frac{3}{2} + 5C_{11}) \backslash B(x_0, \frac{7}{2}), \\ \varphi(x) = w(x), & \text{for } x \in M \backslash B(x_0, \frac{3}{2} + 5C_{11}). \end{cases}
$$

By the definition it is easy to see that $\varphi(x) \in C^\infty(M)$. Since C_{11} depends only on n and k_0, combining (87), (88), (89), (90), (91) and (92) we know that there exists a constant $0 < C_3(n, k_0) < +\infty$ depending only on n and k_0 such that

$$
\text{(93)} \qquad \begin{cases} \frac{1}{C_3}[1 + \gamma(x, x_0)] \le \varphi(x) \le C_3[1 + \gamma(x, x_0)], & \\ |\nabla \varphi(x)| \le C_3, & \forall\ x \in M. \\ |\Delta \varphi(x)| \le C_3, & \end{cases}
$$

Thus we have completed the proof of Theorem 3.1.

Corollary 3.4. *Suppose $(M, g_{ij}(x))$ is an n–dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature:*

$$
\text{(94)} \qquad R_{ij}(x) \ge 0, \quad \forall\ x \in M.
$$

Then there exists a constant $0 < C_{12}(n) < +\infty$ depending only on n such that for any fixed point $x_0 \in M$, there exists a smooth function $\varphi(x) \in C^\infty(M)$ such that

$$
\text{(95)} \qquad \begin{cases} \frac{1}{C_{12}}[1 + \gamma(x, x_0)] \le \varphi(x) \le C_{12}[1 + \gamma(x, x_0)], & \\ |\nabla \varphi(x)| \le C_{12}, & \forall\ x \in M. \\ |\Delta \varphi(x)| \le C_{12}, & \end{cases}
$$

Proof. We let $k_0 = 0$ in (3). Then from Theorem 3.1 we know that the corollary is true. q.e.d.

More generally, we have

Theorem 3.5. *Suppose $(M, g_{ij}(x))$ is an n–dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature:*

$$
\text{(96)} \qquad R_{ij}(x) \ge 0, \quad \forall\ x \in M.
$$

Then there exists a constant $0 < C_{13}(n) < +\infty$ *depending only on* n *such that for any fixed point* $x_0 \in M$ *and any number* $0 < a < +\infty$*, there exists a smooth function* $\varphi(x) \in C^\infty(M)$ *such that*

$$(97)\qquad \begin{cases} \frac{1}{C_{13}}[1 + \frac{\gamma(x,x_0)}{a}] \leq \varphi(x) \leq C_{13}[1 + \frac{\gamma(x,x_0)}{a}], \\ |\nabla\varphi(x)| \leq \frac{C_{13}}{a}, \qquad\qquad\qquad \forall\ x \in M. \\ |\Delta\varphi(x)| \leq \frac{C_{13}}{a^2}, \end{cases}$$

Proof. If $a = 1$, Theorem 3.5 follows directly from Corollary 3.4. If $a \neq 1$, we define a new metric on M:

$$(98)\qquad \tilde{g}_{ij}(x) = \frac{1}{a^2} g_{ij}(x), \quad x \in M.$$

Then $\tilde{g}_{ij}(x)$ is still a complete Riemannian metric on M with nonnegative Ricci curvature. Thus from Corollary 3.4 we know that there exists a smooth function $\varphi(x) \in C^\infty(M)$ such that

$$(99)\qquad \begin{cases} \frac{1}{C_{14}}[1 + \tilde{\gamma}(x,x_0)] \leq \varphi(x) \leq C_{14}[1 + \tilde{\gamma}(x,x_0)], \\ |\tilde{\nabla}\varphi(x)| \leq C_{14}, \qquad\qquad\qquad \forall\ x \in M. \\ |\tilde{\Delta}\varphi(x)| \leq C_{14}. \end{cases}$$

Where $0 < C_{14} < +\infty$ is a constant depending only on n, and $\tilde{\gamma}(x,x_0)$, $\tilde{\nabla}$ and $\tilde{\Delta}$ denote the distance between x and x_0, the covariant derivatives and the Laplacian operator respectively, with respect to the metric $\tilde{g}_{ij}$. Combining (98) and (99) hence shows that (97) is true. q.e.d.

If one reads [40] and [41] carefully, one would see that to establish the maximum principle for the solution of Ricci flow on M the key point is to construct a smooth function $\varphi(x) \in C^\infty(M)$ such that

$$(100)\qquad \begin{cases} \frac{1}{C_{15}}[1 + \gamma(x,x_0)] \leq \varphi(x) \leq C_{15}[1 + \gamma(x,x_0)], \\ |\nabla\varphi(x)| \leq C_{15}, \qquad\qquad\qquad \forall\ x \in M, \\ \nabla_i\nabla_j\varphi(x) \leq C_{15} g_{ij}(x), \end{cases}$$

where $0 < C_{15} < +\infty$ is some constant. In this section we want to prove the following result:

Theorem 3.6. *Suppose* $(M, g_{ij}(x))$ *is an* n*–dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor* $\{R_{ijkl}\}$ *satisfying*

$$(101)\qquad |R_{ijkl}|^2 \leq k_0, \quad on\ M,$$

where $0 < k_0 < +\infty$ *is a constant. Then there exists a constant* $0 < C_{16}(n, k_0) < +\infty$ *depending only on* n *and* k_0 *such that for any fixed point* $x_0 \in M$*, there exists a smooth function* $\varphi(x) \in C^\infty(M)$ *such that*

$$\begin{cases} \frac{1}{C_{16}}[1+\gamma(x,x_0)] \leq \varphi(x) \leq C_{16}[1+\gamma(x,x_0)], \\ |\nabla\varphi(x)| \leq C_{16}, & \forall\ x \in M. \\ |\nabla_i\nabla_j\varphi(x)| \leq C_{16}, \end{cases} \tag{102}$$

Proof. By assumption (101) we have

$$\sup_{x\in M} |R_{ij}(x)|^2 \leq n^2 k_0. \tag{103}$$

Thus the Ricci curvature $R_{ij}(x) \geq -n\sqrt{k_0}$ for any $x \in M$. From Theorem 3.1 it follows that there exists a constant $0 < C_{17}(n, k_0) < +\infty$ depending only on n and k_0 such that for any fixed point $x_0 \in M$, there exists a smooth function $\varphi(x) \in C^\infty(M)$ such that

$$\begin{cases} \frac{1}{C_{17}}[1+\gamma(x,x_0)] \leq \varphi(x) \leq C_{17}[1+\gamma(x,x_0)], \\ |\nabla\varphi(x)| \leq C_{17}, & \forall\ x \in M. \\ |\Delta\varphi(x)| \leq C_{17}, \end{cases} \tag{104}$$

Now we want to use the mollifier technique to modify $\varphi(x)$ such that after the modification, $\nabla_i\nabla_j\varphi(x)$ can be bounded by some constant depending only on n and k_0. This mollifier technique was given by Greene–Wu in their paper [19]. We choose

$$\rho_0 = \pi\left(\frac{1}{k_0}\right)^{\frac{1}{4}}. \tag{105}$$

For any point $x \in M$ and any vector $V \in T_xM$, we use T_xM and $\|V\|$ to denote the tangent space of M at x, and the length of V respectively. For any $\gamma > 0$,

$$\widehat{B}_x(0,\gamma) = \{V \in T_xM \,|\, \|V\| < \gamma\} \tag{106}$$

denotes the ball of radius γ in the tangent space T_xM. Since $|R_{ijkl}|^2 \leq k_0$ on M, using the comparison theorem we know that for any point $x \in M$, the exponential map

$$\exp_x\colon \widehat{B}_x(0,\rho_0) \to M \tag{107}$$

is smooth. Now we choose a smooth function $\alpha(t) \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \alpha(t) \equiv 1, & -\infty < t \leq \frac{1}{4}\rho_0, \\ 0 \leq \alpha(t) \leq 1, & \frac{1}{4}\rho_0 \leq t \leq \frac{1}{2}\rho_0, \\ \alpha(t) \equiv 0, & \frac{1}{2}\rho_0 \leq t < +\infty, \end{cases} \tag{108}$$

$$\begin{cases} |\alpha'(t)| \leq \frac{8}{\rho_0}, & -\infty < t < +\infty, \\ |\alpha''(t)| \leq \frac{400}{\rho_0^2}, & -\infty < t < +\infty. \end{cases} \tag{109}$$

We define a new function $\psi(x)$ on M:

$$\psi(x) = \int_{V \in T_x M} \alpha(\|V\|) \cdot \varphi(\exp_x V) dV, \quad \forall\ x \in M. \tag{110}$$

Then as what Greene–Wu did in their paper [19], $\psi(x) \in C^\infty(M)$ is a smooth function and there exists a constant $0 < C_{18}(n, k_0) < +\infty$ depending only on n and k_0 such that

$$\begin{cases} \frac{1}{C_{18}}[1 + \gamma(x, x_0)] \leq \psi(x) \leq C_{18}[1 + \gamma(x, x_0)], & \\ |\nabla \psi(x)| \leq C_{18}, & \forall\ x \in M. \\ |\nabla_i \nabla_j \psi(x)| \leq C_{18}, & \end{cases} \tag{111}$$

Thus we know that Theorem 3.6 is true. q.e.d.

If we use the iteration argument of J. Moser [35] to control $\nabla_i \nabla_j \mathcal{U}(x)$ for the function $\mathcal{U}(x)$ in Lemma 3.3, then by the use of technique (92) we can also construct a smooth function $\varphi(x) \in C^\infty(M)$ such that (102) is true. This is what we did in §3 of [43].

4. Maximum principles on noncompact manifolds

In the previous section, we constructed some smooth exhaustion functions on complete noncompact Riemannian manifolds. In this section, we are going to use these exhaustion functions to establish the maximum principles on complete noncompact manifolds for the solutions of parabolic equations. In this section we always make the following assumption:

Assumption A. Suppose $(M, \tilde{g}_{ij}(x))$ is an n–dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor

$\{\tilde{R}_{ijkl}\}$. Suppose $0 < T$, $k_0 < +\infty$ are some constants and $g_{ij}(x,t) > 0$ is the smooth solution of the evolution equation

$$(1) \qquad \begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), & \text{on } M \times [0,T], \\ g_{ij}(x,0) = \tilde{g}_{ij}(x), & x \in M, \end{cases}$$

and satisfies the following estimate:

$$(2) \qquad \sup_{M \times [0,T]} |R_{ijkl}(x,t)|^2 \leq k_0.$$

Under the stronger assumption that M has positive sectional curvature at time $t = 0$, some maximum principles were established by the author in [41]. Using the similar arguments as what we did in [41] and the exhaustion functions constructed in the previous section one can establish the maximum principles under Assumption A. Since the proofs are basically the same, we omit many details, which can be seen in [41].

Under Assumption A, we use

$$(3) \qquad d\tilde{s}^2 = \tilde{g}_{ij}(x) dx^i dx^j > 0,$$

$$(4) \qquad ds_t^2 = g_{ij}(x,t) dx^i dx^j > 0, \quad 0 \leq t \leq T,$$

to denote the metrics on M, and use $\tilde{\nabla}$ to denote the covariant derivatives with respect to $d\tilde{s}^2$, use ∇ or ∇^t to denote the covariant derivatives with respect to ds_t^2. We use Δ or Δ_t to denote the Laplacian operator of ds_t^2. For any two points $x, y \in M$, we use $\gamma_t(x,y)$ to denote the distance between x and y with respect to metric ds_t^2.

Lemma 4.1. *Under Assumption A, we have*

$$(5) \qquad \begin{aligned} & e^{-2\sqrt{nk_0}t} d\tilde{s}^2 \leq ds_t^2 \leq e^{2\sqrt{nk_0}t} d\tilde{s}^2, \quad 0 \leq t \leq T, \\ & e^{-\sqrt{nk_0}t} \gamma_0(x,y) \leq \gamma_t(x,y) \leq e^{\sqrt{nk_0}t} \gamma_0(x,y), \; x, y \in M. \end{aligned}$$

Proof. This is Lemma 4.1 in the author's [41]. q.e.d.

From Lemma 4.1 it follows that for any $t \in [0,T]$, the metric ds_t^2 is equivalent to the metric $d\tilde{s}^2$. Thus ds_t^2 is also a complete Riemannian metric on M.

Lemma 4.2. *Under Assumption A, for any integers $m \geq 1$, there exist constants $0 < C(n,m) < +\infty$ depending only on n and m such that*

$$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \leq C(n,m)\left[k_0\left(\frac{1}{t}\right)^m + k_0^{\frac{m}{2}+1}\right],\ 0 \leq t \leq T. \tag{6}$$

Proof. This actually is Lemma 2.3. q.e.d.

Lemma 4.3. *Under Assumption A, we have*

$$\int_0^T |\nabla R_{ijkl}(x,t)|dt \leq 2C(n,1)^{\frac{1}{2}}\left[\sqrt{T\cdot k_0} + \frac{T}{2}\cdot k_0^{\frac{3}{4}}\right],\ x \in M, \tag{7}$$

where $C(n,1)$ is the constant in (6).

Proof. Let $m = 1$. By (6) we get

$$\sup_{x\in M} |\nabla R_{ijkl}(x,t)|^2 \leq C(n,1)\cdot\left[\frac{k_0}{t} + k_0^{\frac{3}{2}}\right],\ 0 \leq t \leq T,$$

$$\sup_{x\in M} |\nabla R_{ijkl}(x,t)| \leq C(n,1)^{\frac{1}{2}}\cdot\left[\frac{\sqrt{k_0}}{\sqrt{t}} + k_0^{\frac{3}{4}}\right],\ 0 \leq t \leq T, \tag{8}$$

$$\int_0^T |\nabla R_{ijkl}(x,t)|dt \leq C(n,1)^{\frac{1}{2}}\int_0^T\left[\frac{\sqrt{k_0}}{\sqrt{t}} + k_0^{\frac{3}{4}}\right]dt,\ x \in M.$$

Thus (7) is true. q.e.d.

Lemma 4.4. *Under Assumption A, for any fixed point $x_0 \in M$, there exists a function $\psi(x) \in C^\infty(M)$ such that*

$$\begin{cases} \frac{1}{C_2}[1+\gamma_0(x,x_0)] \leq \psi(x) \leq C_2[1+\gamma_0(x,x_0)], \\ |\widetilde{\nabla}_i\psi(x)| \leq C_2, \\ |\widetilde{\nabla}_i\widetilde{\nabla}_j\psi(x)| \leq C_2, \end{cases} \quad \forall\ x \in M, \tag{9}$$

where $0 < C_2 < +\infty$ depends only on n and k_0.

Proof. By definition we have $\widetilde{R}_{ijkl}(x) \equiv R_{ijkl}(x,0)$. Thus by (2) we get

$$\sup_{x\in M} |\widetilde{R}_{ijkl}(x)|^2 \leq k_0, \tag{10}$$

and Lemma 4.4 follows directly from Theorem 3.6. q.e.d.

Lemma 4.5. *Under Assumption A, suppose $\psi(x) \in C^\infty(M)$ is the function which we obtained in Lemma 4.4. Then there exists a constant $0 < C_3 < +\infty$ depending only on n, k_0 and T such that*

$$\begin{cases} \frac{1}{C_3}[1+\gamma_t(x,x_0)] \le \psi(x) \le C_3[1+\gamma_t(x,x_0)], \\ |\nabla_i^t \psi(x)| \le C_3, \\ |\nabla_i^t \nabla_j^t \psi(x)| \le C_3, \end{cases} \qquad \text{on } M \times [0,T]. \tag{11}$$

Proof. From (5) it follows that

$$e^{-\sqrt{nk_0}T}\gamma_0(x,y) \le \gamma_t(x,y) \le e^{\sqrt{nk_0}T}\gamma_0(x,y), \qquad x, y \in M, \quad 0 \le t \le T, \tag{12}$$

$$e^{-2\sqrt{nk_0}T}\widetilde{g}_{ij}(x) \le g_{ij}(x,t) \le e^{2\sqrt{nk_0}T}\widetilde{g}_{ij}(x), \qquad 0 \le t \le T. \tag{13}$$

Using (9) and (12) we have

$$\frac{1}{C_4}[1+\gamma_t(x,x_0)] \le \psi(x) \le C_4[1+\gamma_t(x,x_0)], \qquad \text{on } M \times [0,T], \tag{14}$$

where $0 < C_4 < +\infty$ depends only on n, k_0 and T. Since $\psi(x)$ is a function,

$$\nabla_i^t \psi(x) = \widetilde{\nabla}_i \psi(x), \qquad \text{on } M \times [0,T], \tag{15}$$

which together with (9) and (13) yields

$$|\nabla_i^t \psi(x)| \le C_5(n, k_0, T), \qquad \text{on } M \times [0,T]. \tag{16}$$

By definition we have

$$\begin{aligned} \widetilde{\nabla}_i \widetilde{\nabla}_j \psi(x) &= \frac{\partial^2 \psi(x)}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x,0)\frac{\partial \psi(x)}{\partial x^k}, \\ \nabla_i^t \nabla_j^t \psi(x) &= \frac{\partial^2 \psi(x)}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x,t)\frac{\partial \psi(x)}{\partial x^k}, \end{aligned} \tag{17}$$

where $\{\Gamma_{ij}^k(x,t)\}$ denote the Christoffel symbols of $g_{ij}(x,t)$. Thus

$$\begin{aligned} \nabla_i^t \nabla_j^t \psi(x) &= \widetilde{\nabla}_i \widetilde{\nabla}_j \psi(x) - [\Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0)]\frac{\partial \psi(x)}{\partial x^k}, \\ \nabla_i^t \nabla_j^t \psi(x) &= \widetilde{\nabla}_i \widetilde{\nabla}_j \psi(x) - [\Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0)] \cdot \widetilde{\nabla}_k \psi(x). \end{aligned} \tag{18}$$

Using (13), Lemma 4.3 and the arguments developed in the proof of Lemma 4.3 in [41] we obtain

$$|\Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0)|^2 \le C_6(n,k_0,T), \quad \text{on } M \times [0,T], \tag{19}$$

which together with (9), (13) and (18) implies

$$|\nabla_i^t \nabla_j^t \psi(x)| \le C_7(n,k_0,T), \quad \text{on } M \times [0,T]. \tag{20}$$

Combining (14), (16) and (20) we know that (11) is true. q.e.d.

Lemma 4.6. *Under Assumption A, for any constant $0 < C_8 < +\infty$, we can find a function $\theta(x,t) \in C^\infty(M \times [0,T])$ and a constant $0 < C_9 < +\infty$ depending only on n, k_0, T and C_8 such that*

$$0 < \theta(x,t) \le 1, \text{ on } M \times [0,T], \tag{21}$$

$$\frac{C_9^{-1}}{1+\gamma_0(x,x_0)} \le \theta(x,t) \le \frac{C_9}{1+\gamma_0(x,x_0)}, \text{ on } M \times [0,T], \tag{22}$$

$$\frac{\partial \theta}{\partial t} \le \Delta\theta - \frac{2|\nabla_p \theta|^2}{\theta} - C_8\theta, \text{ on } M \times [0,T]. \tag{23}$$

Proof. Basically this is the same as what we did in the proof of Lemma 4.4 in [41], the only difference is that we replace the function $\psi(x)$ in Lemma 4.3 of [41] by the function $\psi(x)$ we obtained in Lemma 4.5 of this paper. q.e.d.

Now we can prove the following maximum principle on noncompact manifold M.

Lemma 4.7. *Under Assumption A, suppose $\varphi(x,t)$ is a C^∞ function on $M \times [0,T]$ such that*

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \Delta\varphi + Q(\varphi,x,t), & \text{on } M \times [0,T], \\ |\varphi(x,t)| \le C_{10} < +\infty, & \text{on } M \times [0,T], \\ \varphi(x,0) \le 0, & \text{on } M, \\ Q(\varphi,x,t) \le 0, & \text{if } \varphi \ge 0. \end{cases} \tag{24}$$

Then we have

$$\varphi(x,t) \le 0, \quad \text{on } M \times [0,T]. \tag{25}$$

Proof. Using Lemma 4.6 and the same arguments as what we did in the proof of Lemma 4.5 in [41], we know that Lemma 4.7 is true.

q.e.d.

Theorem 4.8. *Under Assumption A, suppose $\varphi(x,t)$ is a C^∞ function on $M \times [0,T]$ such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta\varphi + C_{11}|\nabla_k \varphi|^2 + Q(\varphi, x, t), & \text{on } M \times [0,T], \\ \varphi(x,t) \le C_{10} < +\infty, & \text{on } M \times [0,T], \\ \varphi(x,0) \le 0, & \text{on } M, \\ Q(\varphi, x, t) \le C_{12}\varphi, & \text{if } \varphi \ge 0, \end{cases} \tag{26}$$

where $0 \le C_{10}, C_{11}, C_{12} < +\infty$ are some constants. Then we have

$$\varphi(x,t) \le 0, \qquad \text{on } M \times [0,T]. \tag{27}$$

Proof. Basically the same as the proof of Theorem 4.6 in [41], the only difference is that we use Lemma 4.7 of this paper instead of Lemma 4.5 in [41]. q.e.d.

Now we are going to establish another kind of maximum principle on M.

Lemma 4.9. *Under Assumption A, for any fixed point $x_0 \in M$ and constants $\varepsilon > 0$, $h \ge 4$, there exist a function $\theta(x) \in C^\infty(M)$ and a constant $0 < C_{13} < +\infty$ depending only on n, k_0 and ε such that*

$$\begin{cases} 0 \le \theta(x) \le 1, & \text{on } M, \\ \theta(x) \equiv 1, & \forall\ x \in B_0(x_0, h), \\ \theta(x) \equiv 0, & \forall\ x \in M \backslash B_0(x_0, 2C_2^2 h), \end{cases} \tag{28}$$

$$\begin{cases} \left| \widetilde{\nabla}_i \left(\frac{1}{\theta(x)} \right) \right| \le \frac{C_{13}}{h} \left(\frac{1}{\theta(x)} \right)^{1+\varepsilon}, & \forall\ x \in \Omega, \\ \left| \widetilde{\nabla}_i \widetilde{\nabla}_j \left(\frac{1}{\theta(x)} \right) \right| \le \frac{C_{13}}{h} \left(\frac{1}{\theta(x)} \right)^{1+\varepsilon}, & \forall\ x \in \Omega, \end{cases} \tag{29}$$

where C_2 is the constant in (9) and

$$\begin{aligned} &B_0(x_0, h) = \{x \in M | \gamma_0(x, x_0) < h\}, \\ &\Omega = \{x \in M | \theta(x) > 0\}. \end{aligned} \tag{30}$$

Proof. From (96), (99) and (101) in §4 of [41] it follows that there exist two functions $\chi(t)$ and $\eta(t)$ such that

$$\begin{cases} \chi(t) \in C^\infty[0, \frac{7}{4}h), \\ \chi(t) \equiv 1, & 0 \le t \le \frac{5}{4}h, \\ \chi(t) \ge 1, & 0 \le t < \frac{7}{4}h, \\ 0 \le \chi'(t) \le \frac{C_{14}}{h}\chi(t)^{1+\varepsilon}, & 0 \le t < \frac{7}{4}h, \\ |\chi''(t)| \le \frac{C_{14}}{h^2}\chi(t)^{1+\varepsilon}, & 0 \le t < \frac{7}{4}h, \end{cases} \tag{31}$$

$$\begin{cases} \eta(t) = \frac{1}{\chi(t)}, & 0 \le t < \frac{7}{4}h, \\ \eta(t) \equiv 0, & \frac{7}{4}h \le t < +\infty, \\ \eta(t) \in C^\infty[0, +\infty), \end{cases} \tag{32}$$

where $0 < C_{14} < +\infty$ depends only on ε. Suppose $\psi(x) \in C^\infty(M)$ is the function which we obtained in Lemma 4.4, we define

$$\theta(x) = \eta\left(\frac{\psi(x)}{C_2}\right), \quad x \in M. \tag{33}$$

Since $h \ge 4$, by (9) we get

$$\begin{cases} \frac{\psi(x)}{C_2} \le \frac{5}{4}h, & \forall\ x \in B_0(x_0, h), \\ \frac{\psi(x)}{C_2} \ge 2h, & \forall\ x \in M \backslash B_0(x_0, 2C_2^2 h). \end{cases} \tag{34}$$

Combining (31), (32) and (34) yields that $\theta(x) \in C^\infty(M)$ and (28) is true. Hence (29) follows from (9), (31) and (33). q.e.d.

Lemma 4.10. *For the function $\theta(x)$ which we obtained in Lemma 4.9, there exists a constant $0 < C_{15} < +\infty$ depending only on n, k_0, ε and T such that*

$$\begin{aligned} \left|\nabla_i^t\left(\frac{1}{\theta(x)}\right)\right| &\le \frac{C_{15}}{h}\left(\frac{1}{\theta(x)}\right)^{1+\varepsilon}, \quad \forall\ x \in \Omega, \\ & \qquad\qquad 0 \le t \le T. \\ \left|\nabla_i^t\nabla_j^t\left(\frac{1}{\theta(x)}\right)\right| &\le \frac{C_{15}}{h}\left(\frac{1}{\theta(x)}\right)^{1+\varepsilon}, \end{aligned} \tag{35}$$

Proof. Using Lemma 4.9 and the arguments which we used in the proof of Lemma 4.5 we know that (35) is true. q.e.d.

Lemma 4.11. *Under Assumption A, suppose there exist constants* $0 < \varepsilon,\ C_{16},\ C_{17} < +\infty$ *and* $\varphi(x,t) \in C^\infty(M \times [0,T])$ *such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta \varphi + Q(\varphi, x, t), & \text{on } M \times [0,T], \\ \varphi(x,0) \le C_{16}, & \text{on } M, \\ Q(\varphi, x, t) \le -C_{17}\varphi^{1+\varepsilon}, & \text{if } \varphi \ge C_{16}. \end{cases} \tag{36}$$

Then

$$\varphi(x,t) \le C_{16}, \qquad \text{on } M \times [0,T]. \tag{37}$$

Proof. Using Lemma 4.10 and the arguments which we used in the proof of Lemma 4.9 in [41] we know that Lemma 4.11 is true. q.e.d.

Lemma 4.12. *Under Assumption A, suppose* $0 < \varepsilon, C_{16}, C_{17}, C_{18} < +\infty$ *are constants and* $\varphi(x,t) \in C^\infty(M \times [0,T])$ *such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta \varphi + Q(\varphi, x, t), & \text{on } M \times [0,T], \\ \varphi(x,0) \le C_{16}, & \text{on } M, \\ Q(\varphi, x, t) \le \frac{C_{18}}{\varphi}|\nabla_i \varphi|^2 - C_{17}\varphi^{1+\varepsilon}, & \text{if } \varphi \ge C_{16}. \end{cases} \tag{38}$$

Then

$$\varphi(x,t) \le C_{16}, \qquad \text{on } M \times [0,T]. \tag{39}$$

Proof. Using Lemma 4.11 and the arguments as we used in the proof of Lemma 4.10 in [41] we know that Lemma 4.12 is true. q.e.d.

Lemma 4.13. *Under Assumption A, suppose*

$$0 < \varepsilon,\ C_{16}, C_{17}, C_{18}, C_{19} < +\infty$$

are constants and $\varphi(x,t) \in C^\infty(M \times [0,T])$ *such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta \varphi + Q(\varphi, x, t), & \text{on } M \times [0,T], \\ \varphi(x,0) \le C_{16}, & \text{on } M, \\ Q(\varphi, x, t) \le \frac{C_{18}}{\varphi}|\nabla_i \varphi|^2 + \psi_i \cdot \nabla_i \varphi & \\ \qquad\qquad -C_{19}|\psi_i|^2\varphi - C_{17}\varphi^{1+\varepsilon}, & \text{if } \varphi \ge C_{16}, \end{cases} \tag{40}$$

where $\{\psi_i\}$ *is a tensor. Then*

$$\varphi(x,t) \le C_{16}, \qquad on\ M \times [0,T]. \tag{41}$$

Proof. The proof follows from Lemma 4.12 and the inequality

$$\psi_i \cdot \nabla_i \varphi - C_{19}|\psi_i|^2\varphi \le \frac{|\nabla_i\varphi|^2}{4C_{19}\varphi}.$$

Theorem 4.14. *Under Assumption A, suppose* $\varphi(x,t) \in C^\infty(M \times [0,T])$ *and* $0 < \varepsilon,\ C_{11}, C_{12}, C_{16}, C_{17}, C_{18}, C_{19} < +\infty$ *are constants such that*

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \Delta\varphi + Q(\varphi,x,t), & on\ M \times [0,T], \\ \varphi(x,0) \le 0, & on\ M, \\ Q(\varphi,x,t) \le C_{11}|\nabla_i\varphi|^2 + C_{12}\varphi, & if\ 0 \le \varphi \le C_{16}, \\ Q(\varphi,x,t) \le \frac{C_{18}}{\varphi}|\nabla_i\varphi|^2 + \psi_i\cdot\nabla_i\varphi & \\ \qquad\qquad -C_{19}|\psi_i|^2\varphi - C_{17}\varphi^{1+\varepsilon}, & if\ \varphi \ge C_{16}, \end{cases} \tag{42}$$

where $\{\psi_i\}$ *is a tensor. Then*

$$\varphi(x,t) \le 0, \qquad on\ M \times [0,T]. \tag{43}$$

Proof. From Lemma 4.13 it follows that

$$\varphi(x,t) \le C_{16}, \qquad \text{on } M \times [0,T].$$

Using Theorem 4.8 we thus complete the proof. q.e.d.

5. Preserving the Kählerity of the metrics

Suppose $g_{ij}(x,t) > 0$ is the smooth solution of the evolution equation

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \qquad \text{on } M \times [0,T]. \tag{1}$$

In this section we want to show that if $g_{ij}(x,0)$ is a Kähler metric on M, then $g_{ij}(x,t)$ are also Kähler metrics for any $t \in [0,T]$. To prove this statement we need to use the maximum principles established in the previous section.

Theorem 5.1. *Under Assumption A of §4, if M is a complex manifold and $\widetilde{g}_{ij}(x)$ is a Kähler metric on M, then $g_{ij}(x,t)$ are also Kähler metrics for any $t \in [0,T]$.*

Proof. Since M is a complex manifold, we suppose that M has complex dimension n, so that M is a real $2n$–dimensional noncompact manifold. Suppose $z = \{z^1, z^2, \dots, z^n\}$ is the local holomorphic coordinate on M, and

$$\begin{cases} z^k = x^k + \sqrt{-1}x^{k+n}, \\ \\ x^k \in \mathbb{R}, x^{k+n} \in \mathbb{R}, \end{cases} \qquad k = 1, 2, \dots, n. \tag{2}$$

Then $x = \{x^1, x^2, \dots, x^{2n}\}$ is the local real coordinate on M. We use $i, j, k, l, \dots$ to denote the indices corresponding to real vectors or real covectors, $\alpha, \beta, \gamma, \delta, \dots$ the indices corresponding to holomorphic vectors or holomorphic covectors, $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}, \dots$ the indices corresponding to antiholomorphic vectors or antiholomorphic covectors, and $A, B, C, D, \dots$ to denote both $\alpha, \beta, \gamma, \delta, \dots$ and $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}, \dots$.

As a real $2n$–dimensional Riemannian manifold, M has real tangent space $T_{\mathbb{R}}M$ and real cotangent space $T^*_{\mathbb{R}}M$:

$$T_{\mathbb{R}}M = \bigoplus_{i=1}^{2n} \mathbb{R} \cdot \frac{\partial}{\partial x^i}, \tag{3}$$

$$T^*_{\mathbb{R}}M = \bigoplus_{i=1}^{2n} \mathbb{R} \cdot dx^i. \tag{4}$$

If we complexify $T_{\mathbb{R}}M$ and $T^*_{\mathbb{R}}M$, we get the complex tangent space $T_{\mathbb{C}}M$ and complex cotangent space $T^*_{\mathbb{C}}M$ of M as a complex manifold:

$$\begin{aligned} T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes \mathbb{C} &= \bigoplus_{i=1}^{2n} \mathbb{C} \cdot \frac{\partial}{\partial x^i} = \bigoplus_{A} \mathbb{C}\frac{\partial}{\partial z^A} \\ &= \bigoplus_{\alpha} \mathbb{C}\frac{\partial}{\partial z^\alpha} \bigoplus_{\beta} \mathbb{C}\frac{\partial}{\partial \overline{z}^\beta}, \end{aligned} \tag{5}$$

$$T^*_{\mathbb{C}}M = T^*_{\mathbb{R}}M \otimes \mathbb{C} = \bigoplus_{i=1}^{2n} \mathbb{C}\cdot dx^i \qquad (6)$$
$$= \bigoplus_{A} \mathbb{C}\cdot dz^A = \bigoplus_{\alpha} \mathbb{C}\cdot dz^\alpha \bigoplus_{\beta} \mathbb{C}\cdot d\overline{z}^\beta,$$

where

$$\begin{cases} \frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial x^{\alpha+n}}\right), \\ \frac{\partial}{\partial \overline{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1}\frac{\partial}{\partial x^{\alpha+n}}\right), \end{cases} \qquad (7)$$

$$\begin{cases} dz^\alpha = dx^\alpha + \sqrt{-1}dx^{\alpha+n}, \\ d\overline{z}^\alpha = dx^\alpha - \sqrt{-1}dx^{\alpha+n}. \end{cases} \qquad (8)$$

If we denote

$$T^{(1,0)}M = \bigoplus_{\alpha} \mathbb{C}\frac{\partial}{\partial z^\alpha}, \quad T^{(0,1)}M = \bigoplus_{\alpha} \mathbb{C}\frac{\partial}{\partial \overline{z}^\alpha}, \qquad (9)$$

$$T^{*(1,0)}M = \bigoplus_{\alpha} \mathbb{C}\cdot dz^\alpha, \quad T^{*(0,1)}M = \bigoplus_{\alpha} \mathbb{C}\cdot d\overline{z}^\alpha, \qquad (10)$$

then we have the following decompositions:

$$T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M, \qquad (11)$$

$$T^*_{\mathbb{C}}M = T^{*(1,0)}M \oplus T^{*(0,1)}M. \qquad (12)$$

Under Assumption A of §4, we let

$$ds_t^2 = g_{ij}(x,t)dx^i dx^j > 0, \qquad \text{for } 0 \le t \le T. \qquad (13)$$

Using (8) we can write ds_t^2 in terms of complex coordinates on M as follows:

$$\begin{aligned} ds_t^2 =& g_{AB}(z,t)dz^A dz^B \\ =& g_{\alpha\beta}(z,t)dz^\alpha dz^\beta + g_{\alpha\overline{\beta}}(z,t)dz^\alpha d\overline{z}^\beta \\ &+ g_{\overline{\alpha}\beta}(z,t)d\overline{z}^\alpha dz^\beta + g_{\overline{\alpha}\overline{\beta}}(z,t)d\overline{z}^\alpha d\overline{z}^\beta, \quad 0 \le t \le T. \end{aligned} \qquad (14)$$

Since (14) comes from (13), it is easy to see that the following property is true:

$$\begin{cases} \overline{g_{\alpha\beta}(z,t)} = g_{\overline{\alpha}\overline{\beta}}(z,t), \\ \\ \overline{g_{\overline{\alpha}\beta}(z,t)} = g_{\alpha\overline{\beta}}(z,t), \end{cases} \qquad \text{on } M\times[0,T], \qquad (15)$$

which can be simply written as

$$\overline{g_{AB}(z,t)} = g_{\overline{AB}}(z,t), \quad \text{on } M \times [0,T], \tag{16}$$

where we have denoted

$$\begin{cases} \overline{A} = \overline{\alpha}, & \text{if } A = \alpha, \\ \overline{A} = \alpha, & \text{if } A = \overline{\alpha}. \end{cases} \tag{17}$$

By the definition of Kähler metric, ds_t^2 is a Kähler metric if and only if

$$\begin{cases} g_{\alpha\beta}(z,t) \equiv 0, \ g_{\overline{\alpha}\overline{\beta}}(z,t) \equiv 0, \\ \frac{\partial g_{\alpha\overline{\beta}}(z,t)}{\partial z^\gamma} \equiv \frac{\partial g_{\gamma\overline{\beta}}(z,t)}{\partial z^\alpha}, \end{cases} \quad \forall\ z \in M. \tag{18}$$

Similar to $(g^{ij}) = (g_{ij})^{-1}$ in the case of real coordinate, in complex coordinates case we define

$$(g^{AB}) = (g_{AB})^{-1}. \tag{19}$$

The Riemannian curvature tensor $\{R_{ijkl}(x,t)\}$ can also be extended linearly uniquely to $T_{\mathbb{C}}M$ from $T_{\mathbb{R}}M$, thus we get a 4–tensor $\{R_{ABCD}(z,t)\}$ on $T_{\mathbb{C}}M$. The new curvature tensor $\{R_{ABCD}(z,t)\}$ has the same properties as $\{R_{ijkl}(x,t)\}$:

$$\begin{cases} R_{ABCD} = -R_{BACD} = -R_{ABDC} = R_{CDAB}, \\ R_{ABCD} + R_{BCAD} + R_{CABD} = 0, \\ \nabla_E R_{ABCD} + \nabla_A R_{BECD} + \nabla_B R_{EACD} = 0. \end{cases} \tag{20}$$

Similar to (16) we still have

$$\overline{R_{ABCD}(z,t)} = R_{\overline{A}\ \overline{B}\ \overline{C}\ \overline{D}}(z,t), \quad \text{on } M \times [0,T]. \tag{21}$$

We can also define

$$R_{AB}(z,t) = g^{CD}(z,t) \cdot R_{ACBD}(z,t), \quad \text{on } M \times [0,T], \tag{22}$$

$$R(z,t) = g^{AB}(z,t) \cdot R_{AB}(z,t), \quad \text{on } M \times [0,T]. \tag{23}$$

It is easy to see that $\{R_{AB}(z,t)\}$ is also the linear extension of $\{R_{ij}(x,t)\}$ from $T_{\mathbb{R}}M$ to $T_{\mathbb{C}}M$. Since $g_{ij}(x,t)$ is the solution of evolution equation (1) on $M \times [0,T]$, we have

$$\frac{\partial}{\partial t} g_{AB}(z,t) = -2R_{AB}(z,t), \quad \text{on } M \times [0,T]. \tag{24}$$

For the evolution of the curvature tensor, we have

Lemma 5.2. *Suppose $g_{AB}(z,t)$ satisfy (24) on $M \times [0,T]$, then we have*

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ABCD} =& \Delta R_{ABCD} + 2(B_{ABCD} - B_{ABDC} - B_{ADBC} \\
&+ B_{ACBD}) - g^{EF}(R_{EBCD}R_{FA} + R_{AECD}R_{FB} \\
&+ R_{ABED}R_{FC} + R_{ABCE}R_{FD}),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{AB} =& \Delta R_{AB} + 2g^{CD}g^{EF}R_{CAEB}R_{DF} \\
&- 2g^{CD}R_{AC}R_{BD},
\end{aligned} \tag{26}
$$

$$
\frac{\partial}{\partial t} R = \Delta R + 2g^{AB}g^{CD}R_{AC}R_{BD}, \tag{27}
$$

where $B_{ABCD} = g^{EF}g^{GH}R_{EAGB}R_{FCHD}$.

Proof. Since $g_{ij}(x,t)$ satisfy evolution equation (1), from Theorem 7.1, Corollary 7.3 and Corollary 7.5 in R.S. Hamilton [22] we have

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ijkl} =& \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\
&- g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} \\
&+ R_{ijpl}R_{qk} + R_{ijkp}R_{ql}),
\end{aligned} \tag{28}
$$

$$
\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2g^{pr}g^{qs}R_{piqj}R_{rs} - 2g^{pq}R_{pi}R_{qj}, \tag{29}
$$

$$
\frac{\partial}{\partial t} R = \Delta R + 2g^{ij}g^{kl}R_{ik}R_{jl}, \tag{30}
$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$. Writing (28), (29) and (30) in terms of complex coordinates, we know that (25), (26) and (27) are true. q.e.d.

Using Bianchi's Identity (20), it is easy to show that

$$
\begin{cases}
B_{ABDC} - B_{ABCD} = g^{EF}g^{GH}R_{EABG}R_{FHCD}, \\
B_{ABCD} = B_{BADC} = B_{CDAB},
\end{cases} \tag{31}
$$

which together with (25) yield

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ABCD} =& \Delta R_{ABCD} - 2g^{EF}g^{GH}R_{EABG}R_{FHCD} \\
&- 2g^{EF}g^{GH}R_{EAGD}R_{FBHC} + 2g^{EF}g^{GH}R_{EAGC}R_{FBHD} \\
&- g^{EF}(R_{EBCD}R_{FA} + R_{AECD}R_{FB} \\
&+ R_{ABED}R_{FC} + R_{ABCE}R_{FD}).
\end{aligned} \tag{32}
$$

By the definition of $\{R_{ABCD}(z,t)\}$ we obtain

$$|R_{ABCD}(z,t)|^2 = |R_{ijkl}(z,t)|^2, \quad \text{on } M\times[0,T], \tag{33}$$

where

$$\begin{cases} |R_{ijkl}(z,t)|^2 = g^{ip}g^{jq}g^{k\gamma}g^{ls}R_{ijkl}R_{pq\gamma s}, \\ |R_{ABCD}(z,t)|^2 = g^{AE}g^{BF}g^{CG}g^{DH}R_{ABCD}R_{EFGH}. \end{cases} \tag{34}$$

Thus under Assumption A of §4, we have

$$\sup_{M\times[0,T]} |R_{ABCD}(z,t)|^2 \le k_0. \tag{35}$$

To avoid the complicated computation on the change of the metrics $g_{AB}(z,t)$ among the proof of Theorem 5.1, we use the abstract tangent vector bundle method which was originally derived by R.S. Hamilton in [23]. We pick an abstract vector bundle V which is isomorphic to the complex tangent bundle $T_{\mathbb{C}}M$ defined by (5), but with a fixed metric $\widetilde{g}_{AB}$ on the fibers of V. We choose an isometry $\mathcal{U} = \{\mathcal{U}_B^A\}$ between V and $T_{\mathbb{C}}M$ at time $t=0$, and we let the isometry $\mathcal{U}$ evolve by the equation

$$\frac{\partial}{\partial t}\mathcal{U}_B^A = g^{AC}R_{CD}\mathcal{U}_B^D, \quad 0\le t\le T, \tag{36}$$

where g^{AC} and R_{CD} are defined by (19) and (22) respectively. Then the pull–back metrics

$$\widetilde{g}_{AB}(z,t) = g_{CD}(z,t)\cdot\mathcal{U}_A^C(z,t)\cdot\mathcal{U}_B^D(z,t) \tag{37}$$

remain constant in time, it is easy to see that

$$\frac{\partial}{\partial t}\widetilde{g}_{AB}(z,t)\equiv 0, \quad 0\le t\le T, \tag{38}$$

and $\mathcal{U}$ remains an isometry between the varying metric g_{AB} on $T_{\mathbb{C}}M$ and the fixed metric $\widetilde{g}_{AB}$ on V. We use $\mathcal{U}$ to pull the curvature tensor on $T_{\mathbb{C}}M$ back to V:

$$\widetilde{R}_{ABCD}(z,t) = R_{EFGH}\cdot\mathcal{U}_A^E\mathcal{U}_B^F\mathcal{U}_C^G\mathcal{U}_D^H, \quad 0\le t\le T. \tag{39}$$

We can also pull back the Levi–Civita connection $\Gamma = \{\Gamma_{AB}^C\}$ on $T_{\mathbb{C}}M$ to get a connection $\widetilde{\Gamma} = \{\widetilde{\Gamma}_{AB}^C\}$ on V, the covariant derivative of a section $\omega = \{\omega^A\}$ of V is given by

$$\nabla_B\omega^A = \frac{\partial\omega^A}{\partial z^B} + \widetilde{\Gamma}_{BC}^A\omega^C. \tag{40}$$

Moreover, we can take the covariant derivatives of any tensors of V and $T_{\mathbb{C}}M$. In particular we have

$$\nabla_A \mathcal{U}_C^B \equiv 0, \quad \nabla_A \widetilde{g}_{BC} \equiv 0, \quad 0 \le t \le T. \tag{41}$$

We can also define the Laplacian operator

$$\Delta \widetilde{R}_{ABCD} = g^{EF} \nabla_E \nabla_F \widetilde{R}_{ABCD} \tag{42}$$

to be the trace of the second order covariant derivatives. Similar to (32) it is easy to show that

$$\begin{aligned} \frac{\partial}{\partial t} \widetilde{R}_{ABCD} =& \Delta \widetilde{R}_{ABCD} - 2\widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EABG}\widetilde{R}_{FHCD} \\ & - 2\widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EAGD}\widetilde{R}_{FBHC} + 2\widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EAGC}\widetilde{R}_{FBHD}, \end{aligned} \tag{43}$$

where

$$(\widetilde{g}^{AB}) = (\widetilde{g}_{AB})^{-1}. \tag{44}$$

For the details of this technique, one can see Hamilton [23].

By the definition of $\{\widetilde{R}_{ABCD}\}$ we have

$$|\widetilde{R}_{ABCD}(z,t)|^2 \equiv |R_{ABCD}(z,t)|^2, \quad \text{on } M \times [0,T], \tag{45}$$

where

$$|\widetilde{R}_{ABCD}(z,t)|^2 = \widetilde{g}^{AE}\widetilde{g}^{BF}\widetilde{g}^{CG}\widetilde{g}^{DH}\widetilde{R}_{ABCD}\widetilde{R}_{EFGH}. \tag{46}$$

Now we define a function on $M \times [0,T]$:

$$\begin{aligned} \varphi(z,t) =& \widetilde{g}^{\alpha\overline{\xi}}\widetilde{g}^{\beta\overline{\zeta}}\widetilde{g}^{\gamma\overline{\sigma}}\widetilde{g}^{\delta\overline{\eta}}\widetilde{R}_{\alpha\beta\gamma\delta}\widetilde{R}_{\overline{\xi}\overline{\zeta}\overline{\sigma}\overline{\eta}} + \widetilde{g}^{\overline{\alpha}\xi}\widetilde{g}^{\overline{\beta}\zeta}\widetilde{g}^{\gamma\overline{\sigma}}\widetilde{g}^{\delta\overline{\eta}}\widetilde{R}_{\overline{\alpha}\overline{\beta}\gamma\delta}\widetilde{R}_{\xi\zeta\overline{\sigma}\overline{\eta}} \\ & + \widetilde{g}^{\overline{\alpha}\xi}\widetilde{g}^{\beta\overline{\zeta}}\widetilde{g}^{\gamma\overline{\sigma}}\widetilde{g}^{\delta\overline{\eta}}\widetilde{R}_{\overline{\alpha}\beta\gamma\delta}\widetilde{R}_{\xi\overline{\zeta}\overline{\sigma}\overline{\eta}} \\ & + \widetilde{g}^{\alpha\overline{\xi}}\widetilde{g}^{\overline{\beta}\zeta}\widetilde{g}^{\gamma\overline{\sigma}}\widetilde{g}^{\delta\overline{\eta}}\widetilde{R}_{\alpha\overline{\beta}\gamma\delta}\widetilde{R}_{\overline{\xi}\zeta\overline{\sigma}\overline{\eta}}, \quad \text{on } M \times [0,T]. \end{aligned} \tag{47}$$

It is easy to see that $\varphi(z,t) \in C^\infty(M \times [0,T])$ is a well defined smooth function and is independent of the choice of the coordinate $\{z^\alpha\}$ on M.

By the hypothesis of Theorem 5.1, the metric $g_{AB}(z,t)$ is Kähler at time $t = 0$, i.e., $g_{AB}(z,0)$ is a Kähler metric. Thus by definition $\widetilde{g}_{AB}(z,0)$ is a Kähler metric, and from (18) it follows that

$$\widetilde{g}_{\alpha\beta}(z,0) \equiv 0, \quad \widetilde{g}_{\overline{\alpha}\overline{\beta}}(z,0) \equiv 0, \quad \forall\ z \in M. \tag{48}$$

By (38) we obtain

$$\widetilde{g}_{AB}(z,t) \equiv \widetilde{g}_{AB}(z,0), \quad z \in M, \ 0 \le t \le T, \tag{49}$$

which together with (48) yield

$$\widetilde{g}_{\alpha\beta}(z,t) \equiv 0, \quad \widetilde{g}_{\overline{\alpha}\overline{\beta}}(z,t) \equiv 0, \quad \text{on } M \times [0,T]. \tag{50}$$

For any point $z \in M$, from (48) we know that there exists a local holomorphic coordinate $\{z^\alpha\}$ such that

$$\widetilde{g}_{\alpha\overline{\beta}}(z,0) = \delta_{\alpha\beta} \tag{51}$$

at one special point z. Using (49) we get

$$\widetilde{g}_{\alpha\overline{\beta}}(z,t) = \delta_{\alpha\beta}, \quad 0 \le t \le T. \tag{52}$$

Since $(\widetilde{g}^{AB}) = (\widetilde{g}_{AB})^{-1}$, combining (50) and (52) implies

$$\begin{cases} \widetilde{g}^{\alpha\beta}(z,t) = 0, \ \widetilde{g}^{\overline{\alpha}\overline{\beta}}(z,t) = 0, \\ \widetilde{g}^{\alpha\overline{\beta}}(z,t) = \delta_{\alpha\beta}. \end{cases} \tag{53}$$

Similar to (16) and (21) we also have

$$\overline{\widetilde{g}_{AB}(z,t)} = \widetilde{g}_{\overline{A}\ \overline{B}}(z,t), \quad \text{on } M \times [0,T], \tag{54}$$

$$\overline{\widetilde{R}_{ABCD}(z,t)} = \widetilde{R}_{\overline{A}\ \overline{B}\ \overline{C}\ \overline{D}}(z,t), \quad \text{on } M \times [0,T]. \tag{55}$$

In the following computation we always assume that the local coordinate $\{z^\alpha\}$ satisfies (51) at one point. Combining (35) and (45) yields

$$|\widetilde{R}_{ABCD}(z,t)|^2 \le k_0, \quad \text{on } M \times [0,T]. \tag{56}$$

Thus by (53) we get

$$\sum_{A,B,C,D} \widetilde{R}_{ABCD} \cdot \overline{\widetilde{R}_{ABCD}} \le k_0. \tag{57}$$

From (47) and (53) it follows that

$$\begin{aligned} \varphi(z,t) &= \sum_{\alpha,\beta,\gamma,\delta} \left\{ |\widetilde{R}_{\alpha\beta\gamma\delta}|^2 + |\widetilde{R}_{\overline{\alpha}\overline{\beta}\gamma\delta}|^2 + |\widetilde{R}_{\overline{\alpha}\beta\gamma\delta}|^2 + |\widetilde{R}_{\alpha\overline{\beta}\gamma\delta}|^2 \right\} \\ &= \sum_{A,B,\gamma,\delta} |\widetilde{R}_{AB\gamma\delta}|^2. \end{aligned} \tag{58}$$

Thus

$$\varphi(z,t) \geq 0, \quad \text{on } M \times [0,T], \tag{59}$$

and $\varphi(z,t)=0$ if and only if

$$\widetilde{R}_{AB\gamma\delta}(z,t) = 0, \quad \text{for all } A, B, \gamma, \delta. \tag{60}$$

By (43) we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{R}_{AB\gamma\delta} =& \Delta \widetilde{R}_{AB\gamma\delta} - 2\widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EABG}\widetilde{R}_{FH\gamma\delta} \\ & - 2\widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EAG\delta}\widetilde{R}_{FBH\gamma} \\ & + 2\widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EAG\gamma}\widetilde{R}_{FBH\delta}. \end{aligned} \tag{61}$$

From (53) we still have

$$\begin{aligned} \widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EABG}\widetilde{R}_{FH\gamma\delta} =& \widetilde{R}_{EABG}\widetilde{R}_{\overline{E}\overline{G}\gamma\delta} \\ =& \widetilde{R}_{\alpha AB\beta}\widetilde{R}_{\overline{\alpha}\overline{\beta}\gamma\delta} + \widetilde{R}_{\alpha AB\overline{\beta}}\widetilde{R}_{\overline{\alpha}\beta\gamma\delta} \\ & + \widetilde{R}_{\overline{\alpha}AB\beta}\widetilde{R}_{\alpha\overline{\beta}\gamma\delta} + \widetilde{R}_{\overline{\alpha}AB\overline{\beta}}\widetilde{R}_{\alpha\beta\gamma\delta}, \end{aligned} \tag{62}$$

$$\begin{aligned} \widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EAG\delta}\widetilde{R}_{FBH\gamma} =& \widetilde{R}_{EAG\delta}\widetilde{R}_{\overline{E}B\overline{G}\gamma} \\ =& \widetilde{R}_{\alpha A\beta\delta}\widetilde{R}_{\overline{\alpha}B\overline{\beta}\gamma} + \widetilde{R}_{\alpha A\overline{\beta}\delta}\widetilde{R}_{\overline{\alpha}B\beta\gamma} \\ & + \widetilde{R}_{\overline{\alpha}A\beta\delta}\widetilde{R}_{\alpha B\overline{\beta}\gamma} + \widetilde{R}_{\overline{\alpha}A\overline{\beta}\delta}\widetilde{R}_{\alpha B\beta\gamma}, \end{aligned} \tag{63}$$

$$\begin{aligned} \widetilde{g}^{EF}\widetilde{g}^{GH}\widetilde{R}_{EAG\gamma}\widetilde{R}_{FBH\delta} =& \widetilde{R}_{EAG\gamma}\widetilde{R}_{\overline{E}B\overline{G}\delta} \\ =& \widetilde{R}_{\alpha A\beta\gamma}\widetilde{R}_{\overline{\alpha}B\overline{\beta}\delta} + \widetilde{R}_{\alpha A\overline{\beta}\gamma}\widetilde{R}_{\overline{\alpha}B\beta\delta} \\ & + \widetilde{R}_{\overline{\alpha}A\beta\gamma}\widetilde{R}_{\alpha B\overline{\beta}\delta} + \widetilde{R}_{\overline{\alpha}A\overline{\beta}\gamma}\widetilde{R}_{\alpha B\beta\delta}. \end{aligned} \tag{64}$$

Combining (62), (63) and (64) shows that (61) can be written as

$$\frac{\partial}{\partial t}\widetilde{R}_{AB\gamma\delta} = \Delta \widetilde{R}_{AB\gamma\delta} + \widetilde{R}_{CDEF} * \widetilde{R}_{GH\alpha\beta}, \tag{65}$$

where $\widetilde{R}_{CDEF}$ denote the general terms of the curvature tensor, $\widetilde{R}_{GH\alpha\beta}$ denote those terms on which the third indices and the fourth indices are unbar indices, and $*$ denotes the tensor product and linear combinations. From (38) and (58) it follows that

$$\frac{\partial \varphi}{\partial t} = 2\widetilde{R}_{AB\gamma\delta} \cdot \frac{\partial}{\partial t}\widetilde{R}_{AB\gamma\delta}, \tag{66}$$

which together with (65) implies

$$\begin{aligned} \frac{\partial\varphi}{\partial t} &= 2\widetilde{R}_{AB\gamma\delta}\cdot[\Delta\widetilde{R}_{AB\gamma\delta}+\widetilde{R}_{CDEF}*\widetilde{R}_{GH\alpha\beta}] \\ &= \Delta|\widetilde{R}_{AB\gamma\delta}|^2-2|\nabla\widetilde{R}_{AB\gamma\delta}|^2+2\widetilde{R}_{CDEF}*\widetilde{R}_{GH\alpha\beta}*\widetilde{R}_{AB\gamma\delta} \\ &= \Delta\varphi-2|\nabla\widetilde{R}_{AB\gamma\delta}|^2+2\widetilde{R}_{CDEF}*\widetilde{R}_{GH\alpha\beta}*\widetilde{R}_{AB\gamma\delta}. \end{aligned} \tag{67}$$

It is easy to see that

$$2\widetilde{R}_{CDEF}*\widetilde{R}_{GH\alpha\beta}*\widetilde{R}_{AB\gamma\delta}\leq C(n)\cdot|\widetilde{R}_{CDEF}|\cdot|\widetilde{R}_{AB\gamma\delta}|^2, \tag{68}$$

where $0<C(n)<+\infty$ depends only on n. Combining (56), (58) and (68) yields

$$2\widetilde{R}_{CDEF}*\widetilde{R}_{GH\alpha\beta}*\widetilde{R}_{AB\gamma\delta}\leq C(n)\cdot\sqrt{k_0}\cdot\varphi, \tag{69}$$

which together with (67) imply that

$$\frac{\partial\varphi}{\partial t}\leq\Delta\varphi-2|\nabla\widetilde{R}_{AB\gamma\delta}|^2+C(n)\sqrt{k_0}\cdot\varphi. \tag{70}$$

Finally we have

$$\frac{\partial\varphi(z,t)}{\partial t}\leq\Delta\varphi(z,t)+C(n)\sqrt{k_0}\cdot\varphi(z,t),\quad \text{on } M\times[0,T]. \tag{71}$$

Since by the hypothesis of Theorem 5.1 $\widetilde{g}_{AB}(z,0)$ is a Kähler metric, we obtain

$$\widetilde{R}_{AB\gamma\delta}(z,0)\equiv 0,\quad \forall\ A,B,\gamma,\delta; \tag{72}$$

thus from (58) it follows that

$$\varphi(z,0)\equiv 0,\quad \forall\ z\in M. \tag{73}$$

By (57), (58) and (59) we still have

$$0\leq\varphi(z,t)\leq k_0,\quad \text{on } M\times[0,T]. \tag{74}$$

Combining (71), (73), (74) and using maximum principle Theorem 4.8 we get

$$\varphi(z,t)\leq 0,\quad \text{on } M\times[0,T], \tag{75}$$

which together with (74) implies

$$\varphi(z,t) \equiv 0, \quad \text{on } M \times [0,T]. \tag{76}$$

Thus from (60) and (76) it follows that

$$\widetilde{R}_{AB\gamma\delta}(z,t) \equiv 0, \quad \text{on } M \times [0,T]. \tag{77}$$

By the same reason we have

$$\widetilde{R}_{\alpha\beta AB} \equiv 0,\ \widetilde{R}_{AB\overline{\alpha}\overline{\beta}} \equiv 0,\ \widetilde{R}_{\overline{\alpha}\overline{\beta}AB} \equiv 0, \quad \text{on } M \times [0,T]. \tag{78}$$

Now we define

$$\widetilde{R}_{AB}(z,t) = \widetilde{g}^{CD}(z,t)\widetilde{R}_{ACBD}(z,t). \tag{79}$$

By (39) we have

$$\widetilde{R}_{AB}(z,t) = R_{CD}(z,t) \cdot \mathcal{U}_A^C \cdot \mathcal{U}_B^D, \tag{80}$$

$$R_{CD} = \widetilde{R}_{AB} \cdot \mathcal{V}_C^A \cdot \mathcal{V}_D^B, \quad \text{where } (\mathcal{V}_B^A) = (\mathcal{U}_A^B)^{-1}. \tag{81}$$

From (37) we have

$$\begin{cases} \widetilde{g}^{AB} = g^{CD} \cdot \mathcal{V}_C^A \cdot \mathcal{V}_D^B, \\ g^{CD} = \widetilde{g}^{AB} \cdot \mathcal{U}_A^C \cdot \mathcal{U}_B^D. \end{cases} \tag{82}$$

Combining (81) and (82) shows that (36) can be written as

$$\frac{\partial}{\partial t}\mathcal{U}_B^A = \widetilde{g}^{EF}\widetilde{R}_{FB}\mathcal{U}_E^A. \tag{83}$$

Suppose the coordinate satisfies (53) at one point. Then

$$\frac{\partial}{\partial t}\mathcal{U}_B^A = \widetilde{R}_{\overline{E}B}\mathcal{U}_E^A. \tag{84}$$

By the definition of $\mathcal{U} = \{\mathcal{U}_B^A\}$ one can choose a base of the vector bundle V such that

$$\mathcal{U}_B^A(z,0) \equiv \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{if } A \neq B. \end{cases} \tag{85}$$

From (84) it follows that

$$\frac{\partial}{\partial t}\mathcal{U}_{\overline{\beta}}^{\alpha} = \widetilde{R}_{\overline{E}\overline{\beta}}\mathcal{U}_E^{\alpha} = \widetilde{R}_{\overline{\gamma}\overline{\beta}}\mathcal{U}_{\gamma}^{\alpha} + \widetilde{R}_{\gamma\overline{\beta}}\mathcal{U}_{\overline{\gamma}}^{\alpha}. \tag{86}$$

By (53) and (79) we obtain

$$\begin{aligned} \widetilde{R}_{\alpha\beta}(z,t) &= \widetilde{g}^{CD}\widetilde{R}_{\alpha C\beta D} = \widetilde{R}_{\alpha\overline{D}\beta D} \\ &= \widetilde{R}_{\alpha\overline{\gamma}\beta\gamma} + \widetilde{R}_{\alpha\gamma\beta\overline{\gamma}}, \end{aligned} \tag{87}$$

which together with (77), (78) yields

$$\widetilde{R}_{\alpha\beta}(z,t) \equiv 0, \quad \text{on } M \times [0,T]. \tag{88}$$

Similarly,

$$\widetilde{R}_{\overline{\alpha}\overline{\beta}}(z,t) \equiv 0, \quad \text{on } M \times [0,T], \tag{89}$$

which together with (86) implies

$$\frac{\partial}{\partial t}\mathcal{U}^{\alpha}_{\overline{\beta}} = \widetilde{R}_{\gamma\overline{\beta}}\mathcal{U}^{\alpha}_{\overline{\gamma}}, \quad \forall\ \alpha,\beta. \tag{90}$$

From (85) we have

$$\mathcal{U}^{\alpha}_{\overline{\beta}}(z,0) \equiv 0, \quad \forall\ \alpha,\beta, \tag{91}$$

which together with (90) yields

$$\mathcal{U}^{\alpha}_{\overline{\beta}}(z,t) \equiv 0, \quad \text{on } M \times [0,T]. \tag{92}$$

Similarly,

$$\mathcal{U}^{\overline{\alpha}}_{\beta}(z,t) \equiv 0, \quad \text{on } M \times [0,T]. \tag{93}$$

By (39) we get

$$R_{ABCD}(z,t) = \widetilde{R}_{EFGH}(z,t)\cdot \mathcal{V}^{E}_{A}\mathcal{V}^{F}_{B}\mathcal{V}^{G}_{C}\mathcal{V}^{H}_{D}, \tag{94}$$

where $(\mathcal{V}^{A}_{B}) = (\mathcal{U}^{B}_{A})^{-1}$. From (92), (93) it follows that

$$\mathcal{V}^{\alpha}_{\overline{\beta}}(z,t) \equiv 0,\ \mathcal{V}^{\overline{\alpha}}_{\beta}(z,t) \equiv 0, \quad \text{on } M \times [0,T]. \tag{95}$$

Combining (94) and (95) we know that

$$\begin{aligned} R_{AB\gamma\delta}(z,t) &= \widetilde{R}_{EFGH}(z,t)\cdot \mathcal{V}^{E}_{A}\mathcal{V}^{F}_{B}\mathcal{V}^{G}_{\gamma}\mathcal{V}^{H}_{\delta} \\ &= \widetilde{R}_{EF\alpha\beta}(z,t)\cdot \mathcal{V}^{E}_{A}\mathcal{V}^{F}_{B}\mathcal{V}^{\alpha}_{\gamma}\mathcal{V}^{\beta}_{\delta}, \end{aligned} \tag{96}$$

which together with (77) implies

$$R_{AB\gamma\delta}(z,t)\equiv 0, \quad \text{on } M\times[0,T]. \tag{97}$$

Similarly,

$$\begin{cases} R_{AB\overline{\gamma}\overline{\delta}}(z,t)\equiv 0, \\ R_{\gamma\delta AB}(z,t)\equiv 0, \\ R_{\overline{\gamma}\overline{\delta}AB}(z,t)\equiv 0, \end{cases} \quad \text{on } M\times[0,T]. \tag{98}$$

Combining (81) and (95) we get

$$R_{\alpha\beta}(z,t)=\widetilde{R}_{AB}(z,t)\cdot\mathcal{V}_\alpha^A\mathcal{V}_\beta^B=\widetilde{R}_{\gamma\delta}(z,t)\cdot\mathcal{V}_\alpha^\gamma\mathcal{V}_\beta^\delta,$$

which together with (88) implies

$$R_{\alpha\beta}(z,t)\equiv 0, \quad \text{on } M\times[0,T]. \tag{99}$$

Similarly,

$$R_{\overline{\alpha}\overline{\beta}}(z,t)\equiv 0, \quad \text{on } M\times[0,T]. \tag{100}$$

From (24) and (99) we know that

$$\frac{\partial}{\partial t}g_{\alpha\beta}(z,t)\equiv 0, \quad \text{on } M\times[0,T]. \tag{101}$$

Since $g_{AB}(z,0)$ is a Kähler metric, from (18) it follows that

$$g_{\alpha\beta}(z,0)\equiv 0, \quad z\in M,$$

which together with (101) implies

$$g_{\alpha\beta}(z,t)\equiv 0, \quad \text{on } M\times[0,T]. \tag{102}$$

Similarly,

$$g_{\overline{\alpha}\overline{\beta}}(z,t)\equiv 0, \quad \text{on } M\times[0,T]. \tag{103}$$

Using (97), (98) and (20) we get

$$\begin{cases} R_{\alpha\overline{\beta}\gamma\overline{\delta}}=R_{\gamma\overline{\beta}\alpha\overline{\delta}}=R_{\alpha\overline{\delta}\gamma\overline{\beta}}=R_{\gamma\overline{\delta}\alpha\overline{\beta}}, \\ \nabla_\eta R_{\alpha\overline{\beta}\gamma\overline{\delta}}=\nabla_\alpha R_{\eta\overline{\beta}\gamma\overline{\delta}}=\nabla_\gamma R_{\alpha\overline{\beta}\eta\overline{\delta}}, \\ \nabla_{\overline{\eta}} R_{\alpha\overline{\beta}\gamma\overline{\delta}}=\nabla_{\overline{\beta}} R_{\alpha\overline{\eta}\gamma\overline{\delta}}=\nabla_{\overline{\delta}} R_{\alpha\overline{\beta}\gamma\overline{\eta}}, \end{cases} \quad \text{on } M\times[0,T], \tag{104}$$

which implies that

$$\nabla_\alpha R_{\beta\overline{\gamma}} = \nabla_\beta R_{\alpha\overline{\gamma}}, \quad \text{on } M \times [0,T]. \tag{105}$$

From (24) it follows that

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\frac{\partial g_{\alpha\overline{\beta}}(z,t)}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}(z,t)}{\partial z^\alpha}\right] \\ &\qquad = \frac{\partial}{\partial z^\gamma}\left(\frac{\partial}{\partial t} g_{\alpha\overline{\beta}}\right) - \frac{\partial}{\partial z^\alpha}\left(\frac{\partial}{\partial t} g_{\gamma\overline{\beta}}\right) \\ &\qquad = -2\frac{\partial R_{\alpha\overline{\beta}}}{\partial z^\gamma} + 2\frac{\partial R_{\gamma\overline{\beta}}}{\partial z^\alpha}. \end{aligned} \tag{106}$$

By definition we have

$$\begin{aligned} \nabla_\alpha R_{\gamma\overline{\beta}} &= \frac{\partial R_{\gamma\overline{\beta}}}{\partial z^\alpha} - \Gamma^A_{\alpha\gamma} R_{A\overline{\beta}} - \Gamma^A_{\alpha\overline{\beta}} R_{\gamma A}, \\ \nabla_\gamma R_{\alpha\overline{\beta}} &= \frac{\partial R_{\alpha\overline{\beta}}}{\partial z^\gamma} - \Gamma^A_{\gamma\alpha} R_{A\overline{\beta}} - \Gamma^A_{\gamma\overline{\beta}} R_{\alpha A}, \end{aligned} \tag{107}$$

where

$$\Gamma^C_{AB} = \frac{1}{2} g^{CD}\left\{\frac{\partial g_{DB}}{\partial z^A} + \frac{\partial g_{AD}}{\partial z^B} - \frac{\partial g_{AB}}{\partial z^D}\right\}. \tag{108}$$

Combining (106) and (107) gives

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}}{\partial z^\alpha}\right] \\ &\quad = 2\nabla_\alpha R_{\gamma\overline{\beta}} - 2\nabla_\gamma R_{\alpha\overline{\beta}} + 2\Gamma^A_{\alpha\gamma} R_{A\overline{\beta}} - 2\Gamma^A_{\gamma\alpha} R_{A\overline{\beta}} \\ &\qquad + 2\Gamma^A_{\alpha\overline{\beta}} R_{\gamma A} - 2\Gamma^A_{\gamma\overline{\beta}} R_{\alpha A}, \end{aligned} \tag{109}$$

which together with (105) and the fact that $\Gamma^A_{\alpha\gamma} = \Gamma^A_{\gamma\alpha}$ implies

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}}{\partial z^\alpha}\right] &= 2\Gamma^A_{\alpha\overline{\beta}} R_{\gamma A} - 2\Gamma^A_{\gamma\overline{\beta}} R_{\alpha A} \\ &= 2\Gamma^\delta_{\alpha\overline{\beta}} R_{\gamma\delta} + 2\Gamma^{\overline{\delta}}_{\alpha\overline{\beta}} R_{\gamma\overline{\delta}} - 2\Gamma^\delta_{\gamma\overline{\beta}} R_{\alpha\delta} - 2\Gamma^{\overline{\delta}}_{\gamma\overline{\beta}} R_{\alpha\overline{\delta}}, \end{aligned} \tag{110}$$

which together with (99) yields

$$\frac{\partial}{\partial t}\left[\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}}{\partial z^\alpha}\right] = 2\Gamma^{\overline{\delta}}_{\alpha\overline{\beta}} R_{\gamma\overline{\delta}} - 2\Gamma^{\overline{\delta}}_{\gamma\overline{\beta}} R_{\alpha\overline{\delta}}. \tag{111}$$

From (108) we know that

$$\Gamma^{\overline{\delta}}_{\alpha\overline{\beta}} = \frac{1}{2} g^{\overline{\delta}A} \left\{ \frac{\partial g_{A\overline{\beta}}}{\partial z^\alpha} + \frac{\partial g_{\alpha A}}{\partial \overline{z}^\beta} - \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^A} \right\}. \tag{112}$$

Combining (102), (103) and the fact that $(g^{AB}) = (g_{AB})^{-1}$ we get

$$g^{\alpha\beta}(z,t) \equiv 0, \quad g^{\overline{\alpha}\overline{\beta}}(z,t) \equiv 0, \qquad \text{on } M \times [0,T], \tag{113}$$

which together with (112) implies

$$\Gamma^{\overline{\delta}}_{\alpha\overline{\beta}} = \frac{1}{2} g^{\overline{\delta}\eta} \left\{ \frac{\partial g_{\eta\overline{\beta}}}{\partial z^\alpha} + \frac{\partial g_{\alpha\eta}}{\partial \overline{z}^\beta} - \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\eta} \right\}. \tag{114}$$

From (102) it follows that $\frac{\partial}{\partial \overline{z}^\beta} g_{\alpha\eta} \equiv 0$, so that, in consequence of (114),

$$\Gamma^{\overline{\delta}}_{\alpha\overline{\beta}} = \frac{1}{2} g^{\overline{\delta}\eta} \left\{ \frac{\partial g_{\eta\overline{\beta}}}{\partial z^\alpha} - \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\eta} \right\}. \tag{115}$$

Similarly,

$$\Gamma^{\overline{\delta}}_{\gamma\overline{\beta}} = \frac{1}{2} g^{\overline{\delta}\eta} \left\{ \frac{\partial g_{\eta\overline{\beta}}}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}}{\partial z^\eta} \right\}. \tag{116}$$

Substituting (115) and (116) into (111), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} \left[\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}}{\partial z^\alpha} \right] \\ &= g^{\overline{\delta}\eta} R_{\gamma\overline{\delta}} \left[\frac{\partial g_{\eta\overline{\beta}}}{\partial z^\alpha} - \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^\eta} \right] - g^{\overline{\delta}\eta} R_{\alpha\overline{\delta}} \left[\frac{\partial g_{\eta\overline{\beta}}}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}}{\partial z^\eta} \right]. \end{aligned} \tag{117}$$

Since $g_{AB}(z,0)$ is a Kähler metric, by (18) we have

$$\frac{\partial g_{\alpha\overline{\beta}}(z,0)}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}(z,0)}{\partial z^\alpha} \equiv 0, \qquad \text{on } M, \tag{118}$$

which together with (117) implies

$$\frac{\partial g_{\alpha\overline{\beta}}(z,t)}{\partial z^\gamma} - \frac{\partial g_{\gamma\overline{\beta}}(z,t)}{\partial z^\alpha} \equiv 0, \qquad \text{on } M \times [0,T]. \tag{119}$$

Combining (102), (103) and (119) shows that $g_{AB}(z,t)$ is a Kähler metric for every $t \in [0,T]$; thus we have completed the proof of Theorem 5.1.

As soon as we have proved that the evolution equation (1) preserves the Kählerity of the metrics, we are going to show that evolution equation (1) also preserves the nonnegativity and the positivity of the holomorphic bisectional curvature. The corresponding statements in the compact manifolds case were proved by N. Mok in [33].

Theorem 5.3. *Under Assumption A of §4, if M is a complex manifold and $\widetilde{g}_{ij}(x)$ is a Kähler metric on M with nonnegative holomorphic bisectional curvature, then for any $t \in [0,T]$, $g_{ij}(x,t)$ are also Kähler metrics with nonnegative holomorphic bisectional curvature.*

Proof. From Theorem 5.1 we know that for any $t \in [0,T]$, $g_{ij}(x,t)$ are Kähler metrics on M. Thus by (18), (97), (98), (99) and (100) we have

$$\tag{120} \begin{cases} g_{\alpha\beta}(z,t) \equiv 0, \quad g_{\overline{\alpha}\overline{\beta}}(z,t) \equiv 0, \\ R_{AB\gamma\delta}(z,t) \equiv 0, \quad R_{AB\overline{\gamma}\overline{\delta}}(z,t) \equiv 0, \\ R_{\gamma\delta AB}(z,t) \equiv 0, \quad R_{\overline{\gamma}\overline{\delta}AB}(z,t) \equiv 0, \\ R_{\alpha\beta}(z,t) \equiv 0, \quad R_{\overline{\alpha}\overline{\beta}}(z,t) \equiv 0, \end{cases} \qquad \text{on } M \times [0,T],$$

$$\tag{121} \begin{cases} R_{\alpha\overline{\beta}} = g^{AB} R_{\alpha A\overline{\beta}B} = g^{\gamma\overline{\delta}} R_{\alpha\gamma\overline{\beta}\overline{\delta}} + g^{\overline{\delta}\gamma} R_{\alpha\overline{\delta}\overline{\beta}\gamma} = -g^{\gamma\overline{\delta}} R_{\alpha\overline{\beta}\gamma\overline{\delta}}, \\ R(z,t) = g^{AB} R_{AB} = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}} + g^{\overline{\beta}\alpha} R_{\overline{\beta}\alpha} = 2g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}, \end{cases}$$

which together with (32) imply that

$$\tag{122} \begin{aligned} \frac{\partial}{\partial t} R_{\alpha\overline{\beta}\gamma\overline{\delta}} =& \Delta R_{\alpha\overline{\beta}\gamma\overline{\delta}} - 2g^{\overline{\xi}\zeta} g^{\sigma\overline{\eta}} R_{\overline{\xi}\alpha\overline{\beta}\sigma} R_{\zeta\overline{\eta}\gamma\overline{\delta}} \\ & - 2g^{\overline{\xi}\zeta} g^{\sigma\overline{\eta}} R_{\overline{\xi}\alpha\sigma\overline{\delta}} R_{\zeta\overline{\beta}\overline{\eta}\gamma} + 2g^{\overline{\xi}\zeta} g^{\overline{\sigma}\eta} R_{\overline{\xi}\alpha\overline{\sigma}\gamma} R_{\zeta\overline{\beta}\eta\overline{\delta}} \\ & - g^{\xi\overline{\zeta}} (R_{\xi\overline{\beta}\gamma\overline{\delta}} R_{\alpha\overline{\zeta}} + R_{\alpha\overline{\zeta}\gamma\overline{\delta}} R_{\xi\overline{\beta}} + R_{\alpha\overline{\beta}\xi\overline{\delta}} R_{\gamma\overline{\zeta}} \\ & + R_{\alpha\overline{\beta}\gamma\overline{\zeta}} R_{\xi\overline{\delta}}), \\ \frac{\partial}{\partial t} R_{\alpha\overline{\beta}\gamma\overline{\delta}} =& \Delta R_{\alpha\overline{\beta}\gamma\overline{\delta}} - 2g^{\overline{\xi}\zeta} g^{\sigma\overline{\eta}} R_{\alpha\overline{\beta}\sigma\overline{\xi}} R_{\gamma\overline{\delta}\zeta\overline{\eta}} \\ & - 2g^{\overline{\xi}\zeta} g^{\sigma\overline{\eta}} R_{\alpha\overline{\delta}\sigma\overline{\xi}} R_{\gamma\overline{\beta}\zeta\overline{\eta}} + 2g^{\overline{\xi}\zeta} g^{\overline{\sigma}\eta} R_{\alpha\overline{\xi}\gamma\overline{\sigma}} R_{\zeta\overline{\beta}\eta\overline{\delta}} \\ & - g^{\xi\overline{\zeta}} (R_{\xi\overline{\beta}\gamma\overline{\delta}} R_{\alpha\overline{\zeta}} + R_{\alpha\overline{\zeta}\gamma\overline{\delta}} R_{\xi\overline{\beta}} + R_{\alpha\overline{\beta}\xi\overline{\delta}} R_{\gamma\overline{\zeta}} \\ & + R_{\alpha\overline{\beta}\gamma\overline{\zeta}} R_{\xi\overline{\delta}}), \qquad \text{on } M \times [0,T], \end{aligned}$$

where we have used (104) and the fact that $R_{\overline{\beta}\alpha\gamma\overline{\delta}} = -R_{\alpha\overline{\beta}\gamma\overline{\delta}}$. Suppose we use the abstract vector bundle technique as we did in (36), (37), (40) and (41), and suppose $\{\widetilde{R}_{ABCD}\}$ is defined by (39). Since (53), (77) and (78) are true, from (61) it follows that

$$
\begin{aligned}
\frac{\partial}{\partial t}\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} =&\Delta\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} - 2\widetilde{g}^{\overline{\xi}\zeta}\widetilde{g}^{\sigma\overline{\eta}}\widetilde{R}_{\overline{\xi}\alpha\overline{\beta}\sigma}\widetilde{R}_{\zeta\overline{\eta}\gamma\overline{\delta}} \\
&- 2\widetilde{g}^{\overline{\xi}\zeta}\widetilde{g}^{\sigma\overline{\eta}}\widetilde{R}_{\overline{\xi}\alpha\sigma\overline{\delta}}\widetilde{R}_{\zeta\overline{\beta}\overline{\eta}\gamma} + 2\widetilde{g}^{\overline{\xi}\zeta}\widetilde{g}^{\overline{\sigma}\eta}\widetilde{R}_{\overline{\xi}\alpha\overline{\sigma}\gamma}\widetilde{R}_{\zeta\overline{\beta}\eta\overline{\delta}}, \\
\frac{\partial}{\partial t}\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} =&\Delta\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} - 2\widetilde{g}^{\overline{\xi}\zeta}\widetilde{g}^{\sigma\overline{\eta}}\widetilde{R}_{\alpha\overline{\beta}\sigma\overline{\xi}}\widetilde{R}_{\gamma\overline{\delta}\zeta\overline{\eta}} \\
&- 2\widetilde{g}^{\overline{\xi}\zeta}\widetilde{g}^{\sigma\overline{\eta}}\widetilde{R}_{\alpha\overline{\delta}\sigma\overline{\xi}}\widetilde{R}_{\gamma\overline{\beta}\zeta\overline{\eta}} + 2\widetilde{g}^{\overline{\xi}\zeta}\widetilde{g}^{\overline{\sigma}\eta}\widetilde{R}_{\alpha\overline{\xi}\gamma\overline{\sigma}}\widetilde{R}_{\zeta\overline{\beta}\eta\overline{\delta}}, \\
&\text{on } M\times[0,T].
\end{aligned} \tag{123}
$$

If we choose a local holomorphic coordinate $\{z^\alpha\}$ such that $\widetilde{g}_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ at one point, by (123) we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} =&\Delta\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} - 2\widetilde{R}_{\alpha\overline{\beta}\sigma\overline{\xi}}\widetilde{R}_{\gamma\overline{\delta}\xi\overline{\sigma}} \\
&- 2\widetilde{R}_{\alpha\overline{\delta}\sigma\overline{\xi}}\widetilde{R}_{\gamma\overline{\beta}\xi\overline{\sigma}} + 2\widetilde{R}_{\alpha\overline{\xi}\gamma\overline{\sigma}}\widetilde{R}_{\xi\overline{\beta}\sigma\overline{\delta}},
\end{aligned} \tag{124}
$$

which can be written as

$$
\frac{\partial}{\partial t}\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} = \Delta\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} - Q(\widetilde{Rm})_{\alpha\overline{\beta}\gamma\overline{\delta}}, \quad \text{on } M\times[0,T], \tag{125}
$$

where

$$
\begin{aligned}
Q(\widetilde{Rm})_{\alpha\overline{\beta}\gamma\overline{\delta}} =&2\widetilde{R}_{\alpha\overline{\beta}\sigma\overline{\xi}}\widetilde{R}_{\gamma\overline{\delta}\xi\overline{\sigma}} + 2\widetilde{R}_{\alpha\overline{\delta}\sigma\overline{\xi}}\widetilde{R}_{\gamma\overline{\beta}\xi\overline{\sigma}} \\
&- 2\widetilde{R}_{\alpha\overline{\xi}\gamma\overline{\sigma}}\widetilde{R}_{\xi\overline{\beta}\sigma\overline{\delta}}.
\end{aligned} \tag{126}
$$

By definition we know that the holomorphic bisectional curvature is nonnegative if and only if

$$
-R_{\xi\overline{\xi}\zeta\overline{\zeta}} \geq 0, \quad \forall\ \xi,\zeta\in T^{(1,0)}M; \tag{127}
$$

the holomorphic bisectional curvature is positive if and only if

$$
-R_{\xi\overline{\xi}\zeta\overline{\zeta}} > 0, \quad \text{for any } \xi,\zeta\in T^{(1,0)}M, \quad \xi\neq 0,\zeta\neq 0. \tag{128}
$$

Combining (39), (92), (93), (94) and (95) we have

$$
\begin{cases}
\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}(z,t) = R_{\xi\overline{\zeta}\sigma\overline{\eta}}(z,t)\cdot\mathcal{U}_\alpha^\xi\ \mathcal{U}_{\overline{\beta}}^{\overline{\zeta}}\ \mathcal{U}_\gamma^\sigma\ \mathcal{U}_{\overline{\delta}}^{\overline{\eta}}, \\
R_{\xi\overline{\zeta}\sigma\overline{\eta}}(z,t) = \widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}(z,t)\cdot\mathcal{V}_\xi^\alpha\ \mathcal{V}_{\overline{\zeta}}^{\overline{\beta}}\ \mathcal{V}_\sigma^\gamma\ \mathcal{V}_{\overline{\eta}}^{\overline{\delta}};
\end{cases} \tag{129}
$$

thus (127) and (128) are equivalent to

$$-\widetilde{R}_{\xi\bar{\xi}\zeta\bar{\zeta}} \geq 0, \quad \forall\, \xi, \zeta \in T^{(1,0)}M, \tag{130}$$

$$-\widetilde{R}_{\xi\bar{\xi}\zeta\bar{\zeta}} > 0, \quad \forall\, \xi, \zeta \in T^{(1,0)}M,\ \xi \neq 0, \zeta \neq 0, \tag{131}$$

respectively. Therefore to prove Theorem 5.3 we only need to show that

$$-\widetilde{R}_{\xi\bar{\xi}\zeta\bar{\zeta}}(z,t) \geq 0, \qquad \text{on } M \times [0,T]. \tag{132}$$

For any $(z,t) \in M \times [0,T]$, by definition the holomorphic tangent spaces $T_z^{(1,0)}M$ are independent of t. Now we define the subspace

$$S(z) = \{\xi \in T_z^{(1,0)}M \,|\, \|\xi\|^2 = 1\}, \tag{133}$$

where $\|\ \|^2$ are the norms with respect to the metric $\widetilde{g}_{\alpha\bar{\beta}}(z,t)$. From (49) we know that $S(z)$ are independent of time t inside the abstract vector bundle V. We define a function φ on $M \times [0,T]$ by

$$\varphi(z,t) = \sup\{\theta \in \mathbb{R} \,|\, A_{\xi\bar{\xi}\zeta\bar{\zeta}}(\theta, z, t) \geq 0 \quad \text{for any } \xi, \zeta \in S(z)\}, \tag{134}$$

where the tensor $\{A_{\alpha\bar{\beta}\gamma\bar{\delta}}(\theta, z, t)\}$ is defined by

$$\begin{aligned} A_{\alpha\bar{\beta}\gamma\bar{\delta}}(\theta, z, t) = &- \widetilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}(z,t) - \theta \widetilde{g}_{\alpha\bar{\beta}}(z,t) \cdot \widetilde{g}_{\gamma\bar{\delta}}(z,t) \\ &- \theta \widetilde{g}_{\alpha\bar{\delta}}(z,t) \cdot \widetilde{g}_{\gamma\bar{\beta}}(z,t). \end{aligned} \tag{135}$$

It is easy to see that $\varphi(z,t) \in C^0(M \times [0,T])$ is a continuous function. If we define

$$\begin{aligned} A_{\alpha\bar{\beta}\gamma\bar{\delta}}(z,t) = &- \widetilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}(z,t) \\ &- \varphi(z,t)[\widetilde{g}_{\alpha\bar{\beta}}(z,t) \cdot \widetilde{g}_{\gamma\bar{\delta}}(z,t) + \widetilde{g}_{\alpha\bar{\delta}}(z,t) \cdot \widetilde{g}_{\gamma\bar{\beta}}(z,t)], \end{aligned} \tag{136}$$

then by definition

$$A_{\xi\bar{\xi}\zeta\bar{\zeta}}(z,t) \geq 0, \qquad \text{on } M \times [0,T]. \tag{137}$$

For any fixed $(z,t) \in M \times [0,T]$, since $S(z)$ is a compact subset of $T_z^{(1,0)}M$, combining (134), (135) and (136) shows that there exist $\alpha, \beta \in S(z)$ such that

$$A_{\alpha\bar{\alpha}\beta\bar{\beta}}(z,t) = 0, \tag{138}$$

which together with (137) implies

$$\inf\{A_{\xi\bar{\xi}\zeta\bar{\zeta}}(z,t)|\xi,\zeta\in S(z)\}\equiv 0, \quad \text{on } M\times[0,T]. \tag{139}$$

By Lemma 3.5 in R.S. Hamilton [23] we have

$$\begin{aligned}&\frac{\partial}{\partial t}\inf\{A_{\xi\bar{\xi}\zeta\bar{\zeta}}(z,t)|\xi,\zeta\in S(z)\}\\ &\quad\geq \inf\left\{\frac{\partial}{\partial t}A_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)|\alpha,\beta\in S(z) \text{ such that } A_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)=0\right\},\end{aligned} \tag{140}$$

which together with (139) yields

$$\inf\left\{\frac{\partial}{\partial t}A_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)|\alpha,\beta\in S(z) \text{ such that } A_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)=0\right\}\leq 0. \tag{141}$$

For any fixed $(z,t)\in M\times[0,T]$, since $S(z)$ is compact, from (138) and (141) it follows that there exist $\alpha,\beta\in S(z)$ such that

$$\begin{cases} A_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)=0,\\ \frac{\partial}{\partial t}A_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)\leq 0,\end{cases} \tag{142}$$

which together with (137) implies

$$\Delta A_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)\geq 0. \tag{143}$$

On the other hand, by (38), (136) and (142) we obtain

$$\begin{aligned}&[\widetilde{g}_{\alpha\overline{\alpha}}(z,t)\cdot\widetilde{g}_{\beta\overline{\beta}}(z,t)+\widetilde{g}_{\alpha\overline{\beta}}(z,t)\cdot\widetilde{g}_{\beta\overline{\alpha}}(z,t)]\frac{\partial\varphi(z,t)}{\partial t}\\ &\qquad\geq -\frac{\partial}{\partial t}\widetilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t).\end{aligned} \tag{144}$$

Using (41), (136) and (143) we get

$$\begin{aligned}-\Delta\widetilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t)\geq&[\widetilde{g}_{\alpha\overline{\alpha}}(z,t)\cdot\widetilde{g}_{\beta\overline{\beta}}(z,t)\\ &+\widetilde{g}_{\alpha\overline{\beta}}(z,t)\cdot\widetilde{g}_{\beta\overline{\alpha}}(z,t)]\Delta\varphi(z,t).\end{aligned} \tag{145}$$

Combining (125), (144) and (145) gives

$$\begin{aligned}&[\widetilde{g}_{\alpha\overline{\alpha}}\widetilde{g}_{\beta\overline{\beta}}+\widetilde{g}_{\alpha\overline{\beta}}\cdot\widetilde{g}_{\beta\overline{\alpha}}]\frac{\partial\varphi(z,t)}{\partial t}\\ &\quad\geq[\widetilde{g}_{\alpha\overline{\alpha}}\widetilde{g}_{\beta\overline{\beta}}+\widetilde{g}_{\alpha\overline{\beta}}\cdot\widetilde{g}_{\beta\overline{\alpha}}]\Delta\varphi(z,t)+Q(\widetilde{Rm})_{\alpha\overline{\alpha}\beta\overline{\beta}}.\end{aligned} \tag{146}$$

Since $\alpha, \beta \in S(z)$, we have

$$(147)\qquad \begin{aligned} &\widetilde{g}_{\alpha\overline{\alpha}}(z,t) \cdot \widetilde{g}_{\beta\overline{\beta}}(z,t) + \widetilde{g}_{\alpha\overline{\beta}}(z,t) \cdot \widetilde{g}_{\beta\overline{\alpha}}(z,t) \\ &\quad = 1 + |\widetilde{g}_{\alpha\overline{\beta}}(z,t)|^2 > 0, \end{aligned}$$

which together with (146) yields that

$$(148)\qquad \frac{\partial \varphi(z,t)}{\partial t} \geq \Delta\varphi(z,t) + \frac{1}{1 + |\widetilde{g}_{\alpha\overline{\beta}}(z,t)|^2} Q(\widetilde{Rm})_{\alpha\overline{\alpha}\beta\overline{\beta}}.$$

The function $\varphi(z,t)$ may not be smooth at some points of $M \times [0,T]$, but just as what Hamilton did in [23] we can assume without loss of generality that $\varphi(z,t)$ is smooth while using the maximum principle.

Lemma 5.4. *Suppose $\{A_{\alpha\overline{\beta}\gamma\overline{\delta}}\}$ is a tensor which has the same symmetries as $\{\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}\}$. We let*

$$\begin{aligned} Q(A)_{\alpha\overline{\beta}\gamma\overline{\delta}} =& 2A_{\alpha\overline{\beta}\sigma\overline{\xi}} A_{\gamma\overline{\delta}\xi\overline{\sigma}} + 2A_{\alpha\overline{\delta}\sigma\overline{\xi}} A_{\gamma\overline{\beta}\xi\overline{\sigma}} \\ &- 2A_{\alpha\overline{\xi}\gamma\overline{\sigma}} A_{\xi\overline{\beta}\sigma\overline{\delta}}, \end{aligned}$$

where we assume that $\widetilde{g}_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ at one point. Suppose for a fixed point $z \in M$ we have

$$(149)\qquad \begin{cases} A_{\xi\overline{\xi}\zeta\overline{\zeta}} \geq 0, & \forall\ \xi, \zeta \in T_z^{(1,0)}M, \\ A_{\alpha\overline{\alpha}\beta\overline{\beta}} = 0, & \textit{for some } \alpha, \beta \in T_z^{(1,0)}M. \end{cases}$$

Then

$$(150)\qquad Q(A)_{\alpha\overline{\alpha}\beta\overline{\beta}} \geq 0.$$

Proof. The same as what N. Mok did in [33]. q.e.d.

Now suppose $\{A_{\alpha\overline{\beta}\gamma\overline{\delta}}(z,t)\}$ is defined by (136), and $\alpha, \beta \in S(z)$ satisfy (142). From (137), (142) and Lemma 5.4 it follows that

$$(151)\qquad Q(A)_{\alpha\overline{\alpha}\beta\overline{\beta}} \geq 0.$$

On the other hand, by the definition of $Q(A)_{\alpha\overline{\beta}\gamma\overline{\delta}}$ it is easy to see that

$$(152)\qquad \begin{aligned} Q(A)_{\alpha\overline{\beta}\gamma\overline{\delta}} &= Q(\widetilde{Rm})_{\alpha\overline{\beta}\gamma\overline{\delta}} + \varphi(z,t) * \widetilde{Rm} \\ &\quad + \varphi(z,t)^2 * \widetilde{g} * \widetilde{g}, \end{aligned}$$

where $*$ means the linear combinations of the tensor product. Combining (56), (77) and (78) gives

$$|\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}(z,t)|^2 \leq k_0, \quad \text{on } M \times [0,T]. \tag{153}$$

Thus by the definition of $\varphi(z,t)$ in (134) and (135) we get

$$|\varphi(z,t)| \leq \sqrt{k_0}, \quad \text{on } M \times [0,T], \tag{154}$$

which together with (152) and (153) implies

$$|Q(A)_{\alpha\overline{\beta}\gamma\overline{\delta}} - Q(\widetilde{Rm})_{\alpha\overline{\beta}\gamma\overline{\delta}}| \leq C_3(n,k_0)|\varphi(z,t)|, \tag{155}$$

where $0 < C_3(n,k_0) < +\infty$ depends only on n and k_0. Combining (151) and (155) we obtain

$$Q(\widetilde{Rm})_{\alpha\overline{\alpha}\beta\overline{\beta}} \geq -C_3(n,k_0)|\varphi(z,t)|, \quad \text{on } M \times [0,T], \tag{156}$$

which together with (148) yields

$$\frac{\partial \varphi(z,t)}{\partial t} \geq \Delta\varphi(z,t) - C_3(n,k_0)|\varphi(z,t)|, \quad \text{on } M \times [0,T]. \tag{157}$$

On the other hand, by the assumption of Theorem 5.3 we have

$$-\widetilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,0) \geq 0, \quad \forall\ z \in M; \tag{158}$$

thus from (134) and (135) it follows that

$$\varphi(z,0) \geq 0, \quad \forall\ z \in M. \tag{159}$$

Combining (154), (157), (159) and using Theorem 4.8 we know that

$$\varphi(z,t) \geq 0, \quad \text{on } M \times [0,T], \tag{160}$$

which implies that

$$-\widetilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(z,t) \geq 0, \quad \text{on } M \times [0,T]; \tag{161}$$

thus by the explanation in (132), Theorem 5.3 is true.

Theorem 5.5. *Under Assumption A of §4, if M is a complex manifold and $\widetilde{g}_{i\overline{j}}(x)$ is a Kähler metric on M with positive holomorphic bisectional curvature, then for any $t \in [0,T]$, the metrics $g_{i\overline{j}}(x,t)$ are also Kähler metrics with positive holomorphic bisectional curvature.*

Proof. From Theorem 5.3 it follows that $g_{ij}(x,t)$ are Kähler metrics with nonnegative holomorphic bisectional curvature. Using the local technique as what R.S. Hamilton described in [23] we know that $g_{ij}(x,t)$ actually have positive holomorphic bisectional curvature for any $t \in [0,T]$ provided $\widetilde{g}_{ij}(x)$ has positive holomorphic bisectional curvature.

q.e.d.

6. Controlling the volume element

In this section we want to control the volume element of the solution to the Ricci flow evolution equation. Under the assumptions of Theorem 1.1, the author of this paper derived the techniques which were used to control the volume element of the solution to the Ricci flow equation in his Ph.D. thesis [43] in 1990. Later on we found that with some modifications of the techniques appeared in [43], we can still control the volume element of the solution to the Ricci flow equation under much weaker assumptions than that of Theorem 1.1. In this section we describe the modified version of the techniques appeared in §6 of [43].

We make the following assumption:

Assumption B. Suppose M is a complete noncompact Kähler manifold of complex dimension n with its Kähler metric $\widetilde{g}_{ij}(x) > 0$. Suppose $0 < \theta < 2$, $0 < T$, k_0, Θ_0, $C_1 < +\infty$ are constants and $g_{ij}(x,t) > 0$ is the smooth solution to the Ricci flow equation

$$(1) \qquad \begin{cases} \dfrac{\partial}{\partial t}\, g_{ij}(x,t) = -2R_{ij}(x,t), & \text{on } M \times [0,T], \\ g_{ij}(x,0) = \widetilde{g}_{ij}(x), & \text{on } M, \end{cases}$$

which satisfies the following assumptions:

$$(2) \qquad \text{(i)} \quad 0 \le -R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,0) \le k_0, \quad x \in M,$$

$$(3) \qquad \text{(ii)} \quad \int_{B_0(x_0,\gamma)} R(x,0)dV_0 \le \frac{C_1}{(\gamma+1)^\theta} \cdot \mathrm{Vol}(B_0(x_0,\gamma)), \quad x_0 \in M,\ 0 \le \gamma < +\infty,$$

$$(4) \qquad \text{(iii)} \quad \sup_{M\times[0,T]} |R_{ijkl}(x,t)|^2 \le \Theta_0,$$

where we let

$$(5) \qquad \begin{cases} ds_t^2 = g_{ij}(x,t)dx^i dx^j, \\ d\widetilde{s}^2 = ds_0^2, \end{cases}$$

and use $B_t(x,\gamma)$ to denote the geodesic ball of radius γ and centered at $x \in M$ with respect to ds_t^2, dV_t the volume element of ds_t^2, $\{R_{ijkl}(x,t)\}$ the curvature tensor of ds_t^2, $R(x,t)$ the scalar curvature of ds_t^2, and $\mathrm{Vol}(B_t(x,\gamma))$ the volume of $B_t(x,\gamma)$.

Under Assumption B, since $g_{ij}(x,0)$ is a Kähler metric with non-negative holomorphic bisectional curvature, from Theorem 5.3 it follows that for any $t \in [0,T]$, $g_{ij}(x,t)$ are also Kähler metrics with nonnegative holomorphic bisectional curvature:

$$-R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) \geq 0, \quad \text{on } M \times [0,T]. \tag{6}$$

By (6) we get

$$R_{\alpha\overline{\beta}}(x,t) \geq 0, \quad \text{on } M \times [0,T], \tag{7}$$

$$R(x,t) \geq 0, \quad \text{on } M \times [0,T]. \tag{8}$$

We define a function $F(x,t)$ on $M \times [0,T]$:

$$F(x,t) = \log \frac{\det(g_{\alpha\overline{\beta}}(x,t))}{\det(g_{\alpha\overline{\beta}}(x,0))}. \tag{9}$$

By the definition we have

$$dV_t = e^{F(x,t)} dV_0, \quad \text{on } M \times [0,T], \tag{10}$$

$$\begin{aligned} \frac{\partial F(x,t)}{\partial t} &= g^{\alpha\overline{\beta}}(x,t) \cdot \frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x,t) \\ &= -2 g^{\alpha\overline{\beta}}(x,t) R_{\alpha\overline{\beta}}(x,t), \quad \text{on } M \times [0,T], \end{aligned}$$

$$\frac{\partial}{\partial t} F(x,t) = -R(x,t), \quad \text{on } M \times [0,T], \tag{11}$$

which, together with (10) and (8), yields respectively

$$\frac{\partial}{\partial t} dV_t = -R(x,t) dV_t, \quad \text{on } M \times [0,T], \tag{12}$$

$$\frac{\partial}{\partial t} F(x,t) \leq 0, \quad \text{on } M \times [0,T]. \tag{13}$$

On the other hand, by definition we have

$$F(x,0) \equiv 0, \quad x \in M, \tag{14}$$

which together with (13) implies

$$F(x,t) \le 0, \quad \text{on } M \times [0,T]. \tag{15}$$

What we are going to do in this section is to prove the following theorem:

Theorem 6.1. *Under Assumption B, there exists a constant* $C(n, k_0, \theta, C_1)$ *such that*

$$F(x,t) \ge -C(n,k_0,\theta,C_1) \cdot (t+2)^{\frac{2-\theta}{\theta}}, \quad \text{on } M \times [0,T], \tag{16}$$

where $0 < C(n,k_0,\theta,C_1) < +\infty$ *depends only on* n, k_0, θ *and* C_1*, and is independent of* Θ_0 *and* T*.*

To prove Theorem 6.1 we need to control the volume growth rate of $\widetilde{g}_{ij}(x)$ on M. But for the Kähler manifold $(M, \widetilde{g}_{ij}(x))$ in Assumption B, we do not know what is the volume growth rate of $\widetilde{g}_{ij}(x)$ on M. To resolve this problem, we replace the Kähler manifold $(M, \widetilde{g}_{ij}(x))$ in Assumption B by a new manifold

$$\widehat{M} = M \times \mathbb{C}^2 \tag{17}$$

with the product metric

$$d\widehat{s}^2 = \widetilde{g}_{ij}(x)dx^i dx^j + dw^1 d\overline{w}^1 + dw^2 d\overline{w}^2, \tag{18}$$

where $\widetilde{g}_{ij}(x)dx^i dx^j$ is the Kähler metric on M which satisfies Assumption B, and $dw^1 d\overline{w}^1 + dw^2 d\overline{w}^2$ is the standard flat Kähler metric on $\mathbb{C}^2$.

By definitions (17) and (18) we know that $(\widehat{M}, d\widehat{s}^2)$ is also a complete noncompact Kähler manifold which satisfies:

$$0 \le -\widehat{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(y) \le k_0, \quad \forall\ y \in \widehat{M}, \tag{19}$$

$$\int_{\widehat{B}(y_0,\gamma)} \widehat{R}(y)dy \le \frac{C_3}{(\gamma+1)^{\theta}} \cdot \mathrm{Vol}(\widehat{B}(y_0,\gamma)), \qquad \forall\ y_0 \in \widehat{M},\ 0 \le \gamma < +\infty, \tag{20}$$

where k_0 and θ are the constants in Assumption B, $0 < C_3 < +\infty$ is a constant depending only on the constants n, θ and C_1 in Assumption B, $\widehat{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}$ and $\widehat{R}$ denote the curvature tensor and the scalar curvature of the metric $d\widehat{s}^2$ respectively, and $\widehat{B}(y_0,\gamma)$ denote the geodesic balls of $d\widehat{s}^2$ on $\widehat{M}$.

Moreover, we have

$$\dim_{\mathbb{C}} \widehat{M} = n + 2 \geq 3. \tag{21}$$

Since the Ricci curvature on $\widehat{M}$ is nonnegative, using the volume comparison theorem in [5] we get

$$\frac{\operatorname{Vol}(\widehat{B}(y_0, \gamma_2))}{\operatorname{Vol}(\widehat{B}(y_0, \gamma_1))} \leq \left(\frac{\gamma_2}{\gamma_1}\right)^{2(n+2)}, \ \forall\ y_0 \in \widehat{M},\ 0 < \gamma_1 \leq \gamma_2 < +\infty. \tag{22}$$

Since the factor $\mathbb{C}^2$ is flat, it is easy to see that there exists a constant $0 < C_2 < +\infty$ depending only on n such that

$$\frac{\operatorname{Vol}(\widehat{B}(y_0, \gamma_2))}{\operatorname{Vol}(\widehat{B}(y_0, \gamma_1))} \geq C_2 \left(\frac{\gamma_2}{\gamma_1}\right)^{4}, \ \forall\ y_0 \in \widehat{M},\ 0 < \gamma_1 \leq \gamma_2 < +\infty. \tag{23}$$

Now suppose $g_{ij}(x,t) > 0$ is the solution to the Ricci flow equation (1) in Assumption B. If we let

$$d\widehat{s}_t^2 = g_{ij}(x,t)dx^i dx^j + dw^1 d\overline{w}^1 + dw^2 d\overline{w}^2, \quad \text{on } \widehat{M} \times [0,T], \tag{24}$$

then $d\widehat{s}_t^2$ also satisfies the Ricci flow evolution equation on $\widehat{M}$:

$$\begin{cases} \dfrac{\partial}{\partial t} d\widehat{s}_t^2 = -2 \cdot \operatorname{Ricci}(d\widehat{s}_t^2), & \text{on } \widehat{M} \times [0,T], \\ d\widehat{s}_0^2 = d\widehat{s}^2, & \text{on } \widehat{M}. \end{cases} \tag{25}$$

Thus $(\widehat{M}, d\widehat{s}_t^2)$ on $0 \leq t \leq T$ satisfy the following assumption:

Assumption C. Suppose M is a complete noncompact Kähler manifold of complex dimension n with its Kähler metric $\widetilde{g}_{ij}(x) > 0$. Suppose $0 < \theta < 2,\ 0 < T,\ k_0, \Theta, C_2,\ C_3 < +\infty$ are constants and $g_{ij}(x,t) > 0$ is the smooth solution to the Ricci flow equation

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), & \text{on } M \times [0,T], \\ g_{ij}(x,0) = \widetilde{g}_{ij}(x), & \text{on } M, \end{cases} \tag{26}$$

which satisfies the following assumptions:

(27) (i) $\dim_{\mathbb{C}} M = n \geq 3,$

(28) (ii) $0 \leq -R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,0) \leq k_0, \quad x \in M,$

(29) (iii) $$C_2\left(\frac{\gamma_2}{\gamma_1}\right)^4 \leq \frac{\mathrm{Vol}(B_0(x,\gamma_2))}{\mathrm{Vol}(B_0(x,\gamma_1))} \leq \left(\frac{\gamma_2}{\gamma_1}\right)^{2n},$$
$$x \in M,\ 0 < \gamma_1 \leq \gamma_2 < +\infty,$$

(30) (iv) $$\int_{B_0(x_0,\gamma)} R(x,0)dV_0 \leq \frac{C_3}{(\gamma+1)^\theta}\cdot \mathrm{Vol}(B_0(x_0,\gamma)),$$
$$x_0 \in M,\ 0 \leq \gamma < +\infty,$$

(31) (v) $\sup\limits_{M\times[0,T]} |R_{ijkl}(x,t)|^2 \leq \Theta.$

Since $d\widehat{s}_t^2$ in (24) are product metrics for all $0 \leq t \leq T$, thus if we can prove that $d\widehat{s}_t^2$ satisfy inequality (16), then $g_{ij}(x,t)$ in (24) also satisfy inequality (16). Hence in summary, Theorem 6.1 can be deduced from the following theorem:

Theorem 6.2. *Under Assumption C, there exists a constant* $C(n,k_0,\theta,C_2,C_3)$ *such that*

(32) $$F(x,t) \geq -C(n,k_0,\theta,C_2,C_3)\cdot(t+2)^{\frac{2-\theta}{\theta}}, \quad \text{on } M\times[0,T],$$

where $0 < C(n,k_0,\theta,C_2,C_3) < +\infty$ *depends only on* n, k_0, θ, C_2 *and* C_3, *and is independent of* Θ *and* T.

In the remainder of this section, we always assume that Assumption C holds.

Under Assumption C, since $g_{ij}(x,0)$ is a Kähler metric with nonnegative holomorphic bisectional curvature, from Theorem 5.3 it follows that for any $t \in [0,T]$, $g_{ij}(x,t)$ are also Kähler metrics with nonnegative holomorphic bisectional curvature. It is easy to see that (6), (7), (8), (10), (11), (12), (13), (14) and (15) are still true. Since $R_{ij}(x,t) \geq 0$ on $M\times[0,T]$, using the volume comparison theorem in [5] we have

(33) $$\mathrm{Vol}(B_t(x,\gamma)) \leq C_4(n)\cdot\gamma^{2n},\ x\in M,\ 0\leq\gamma<+\infty,\ 0\leq t\leq T,$$

where $0 < C_4(n) < +\infty$ is a constant depending only on n. Combining (8) and (31) we get

(34) $$0 \leq R(x,t) \leq 4n^2\sqrt{\Theta}, \quad \text{on } M\times[0,T],$$

which together with (11) implies

$$0 \geq \frac{\partial}{\partial t} F(x,t) \geq -4n^2\sqrt{\Theta}, \quad \text{on } M \times [0,T]. \tag{35}$$

Since $F(x,0) \equiv 0$, we thus have

$$0 \geq F(x,t) \geq -4n^2\sqrt{\Theta}t, \quad \text{on } M \times [0,T]. \tag{36}$$

Combining (10) and (36) yields

$$dV_0 \geq dV_t \geq e^{-4n^2\sqrt{\Theta}t} dV_0, \quad \text{on } M \times [0,T]. \tag{37}$$

To prove Theorem 6.2 we need to use the smooth exhaustion functions constructed in §3. From Assumption C and (7) it follows that

$$R_{ij}(x,0) \geq 0, \quad \forall\ x \in M. \tag{38}$$

Suppose $x_0 \in M$ is a fixed point, and $1 \leq a < +\infty$ is a constant to be determined later. Then from Theorem 3.5 we know that there exists a function $\psi(x) \in C^\infty(M)$ such that

$$\begin{cases} 1 + \frac{\gamma_0(x,x_0)}{a} \leq \psi(x) \leq C_5\left[1 + \frac{\gamma_0(x,x_0)}{a}\right], \\ |\widetilde{\nabla}\psi(x)|_0 \leq \frac{C_5}{a}, \qquad \forall\ x \in M, \\ |\Delta_0\psi(x)| \leq \frac{C_5}{a^2}, \end{cases} \tag{39}$$

where $0 < C_5 < +\infty$ is a constant depending only on n, $\gamma_t(x,x_0)$ is the distance between x and x_0 with respect to ds_t^2, $\widetilde{\nabla}$ is the covariant derivatives with respect to ds_0^2, $|\ |_0$ is the norm with respect to ds_0^2, and Δ_t is the Laplacian operator with respect to ds_t^2. We now let

$$\varphi(x) = e^{-\psi(x)}, \quad x \in M. \tag{40}$$

From (39) we have

$$\begin{cases} \varphi(x) \in C^\infty(M), \\ \varphi(x) \leq e^{-[1+\frac{\gamma_0(x,x_0)}{a}]}, \qquad x \in M, \\ \varphi(x) \geq e^{-C_6[1+\frac{\gamma_0(x,x_0)}{a}]}, \quad x \in M, \\ |\widetilde{\nabla}\varphi(x)|_0 \leq \frac{C_6}{a}\varphi(x), \qquad x \in M, \\ |\Delta_0\varphi(x)| \leq \frac{C_6}{a^2}\varphi(x), \qquad x \in M, \end{cases} \tag{41}$$

where $0 < C_6 < +\infty$ is a constant depending only on n.

Lemma 6.3. *There exists a constant* $0 < C_7(n) < +\infty$ *depending only on* n *such that*

$$\int_M \varphi(x)dV_0 \leq C_7(n) \cdot \mathrm{Vol}(B_0(x_0, a)). \tag{42}$$

Also there exists a constant $0 < C_8(n, \theta, C_3) < +\infty$ *depending only on* n, θ *and* C_3 *such that*

$$\int_M R(x, 0)\varphi(x)dV_0 \leq \frac{C_8}{a^\theta} \cdot \mathrm{Vol}(B_0(x_0, a)). \tag{43}$$

Proof. Since (42) is a special case of (43) when $R(x, 0) \equiv 1$, $\theta = 0$ and $C_3 = 1$, we only need to prove (43). From (41) we have

$$\begin{aligned}
&\int_M R(x, 0)\varphi(x)dV_0 \\
&\leq \int_M R(x, 0) \cdot e^{-[1+\frac{\gamma_0(x,x_0)}{a}]}dV_0 \\
&= \int_{B_0(x_0,a)} R(x, 0) \cdot e^{-[1+\frac{1}{a}\gamma_0(x,x_0)]}dV_0 \\
&\quad + \sum_{k=0}^{\infty} \int_{B_0(x_0,2^{k+1}a)\setminus B_0(x_0,2^k a)} R(x, 0) \cdot e^{-[1+\frac{1}{a}\gamma_0(x,x_0)]}dV_0 \\
&\leq \int_{B_0(x_0,a)} R(x, 0)dV_0 \\
&\quad + \sum_{k=0}^{\infty} e^{-2^k} \int_{B_0(x_0,2^{k+1}a)\setminus B_0(x_0,2^k a)} R(x, 0)dV_0 \\
&\leq \int_{B_0(x_0,a)} R(x, 0)dV_0 + \sum_{k=0}^{\infty} e^{-2^k} \int_{B_0(x_0,2^{k+1}a)} R(x, 0)dV_0,
\end{aligned}$$

which together with (30) of Assumption C implies

$$\begin{aligned}
\int_M R(x, 0)\varphi(x)dV_0 \leq& \frac{C_3}{(a+1)^\theta} \cdot \mathrm{Vol}(B_0(x_0, a)) \\
&+ \sum_{k=0}^{\infty} e^{-2^k} \cdot \frac{C_3}{(2^{k+1}a+1)^\theta} \cdot \mathrm{Vol}(B_0(x_0, 2^{k+1}a)) \\
\leq& \frac{C_3}{a^\theta} \cdot \mathrm{Vol}(B_0(x_0, a)) \\
&+ \sum_{k=0}^{\infty} e^{-2^k} \cdot \frac{C_3}{(2^{k+1}a)^\theta} \cdot \mathrm{Vol}(B_0(x_0, 2^{k+1}a)),
\end{aligned}$$

which together with (29) of Assumption C yields

$$\begin{aligned}\int_M R(x,0)\varphi(x)dV_0 \le & \frac{C_3}{a^\theta}\cdot \mathrm{Vol}(B_0(x_0,a)) \\ & + \sum_{k=0}^{\infty} e^{-2^k}\cdot \frac{C_3}{(2^{k+1}a)^\theta}\cdot (2^{k+1})^{2n}\cdot \mathrm{Vol}(B_0(x_0,a)) \\ \le & \frac{C_8(n,\theta,C_3)}{a^\theta}\cdot \mathrm{Vol}(B_0(x_0,a)).\end{aligned} \tag{44}$$

Thus (43) is true. q.e.d.

From (37) it follows that

$$0 < \int_M \varphi(x)dV_t \le \int_M \varphi(x)dV_0, \quad 0\le t\le T, \tag{45}$$

which, together with (42), implies

$$0 < \int_M \varphi(x)dV_t \le C_7\cdot \mathrm{Vol}(B_0(x_0,a)), \quad 0\le t\le T. \tag{46}$$

By (12) we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \varphi(x)dV_t &= \int_M \varphi(x)\frac{\partial}{\partial t}dV_t \\ &= -\int_M \varphi(x)R(x,t)dV_t,\ 0\le t\le T.\end{aligned} \tag{47}$$

From (7) we know that

$$\frac{\partial}{\partial t}g_{\alpha\overline{\beta}}(x,t) = -2R_{\alpha\overline{\beta}}(x,t)\le 0, \quad \text{on } M\times[0,T], \tag{48}$$

$$g_{\alpha\overline{\beta}}(x,t)\le g_{\alpha\overline{\beta}}(x,0), \quad \text{on } M\times[0,T]. \tag{49}$$

Combining (10) and (49) gives

$$\begin{aligned}R(x,t)dV_t &= R(x,t)e^{F(x,t)}dV_0 \\ &= 2g^{\alpha\overline{\beta}}(x,t)R_{\alpha\overline{\beta}}(x,t)\cdot e^{F(x,t)}dV_0 \\ &= 2g^{\alpha\overline{\beta}}(x,t)R_{\alpha\overline{\beta}}(x,t)\cdot \frac{\det(g_{\gamma\overline{\delta}}(x,t))}{\det(g_{\gamma\overline{\delta}}(x,0))}dV_0 \\ &\le 2g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t)dV_0, \quad \text{on } M\times[0,T],\end{aligned} \tag{50}$$

where we have used (7). Substituting (50) into (47) yields

$$\frac{\partial}{\partial t}\int_M \varphi(x)dV_t \geq -2\int_M g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t)\varphi(x)dV_0,\ 0\leq t\leq T. \tag{51}$$

By the definition (9) of $F(x,t)$, we have

$$\frac{\partial^2 F(x,t)}{\partial z^\alpha \partial \overline{z}^\beta} = R_{\alpha\overline{\beta}}(x,0) - R_{\alpha\overline{\beta}}(x,t). \tag{52}$$

Thus

$$\begin{aligned}\Delta_0 F(x,t) =& 2g^{\alpha\overline{\beta}}(x,0)\frac{\partial^2 F(x,t)}{\partial z^\alpha \partial \overline{z}^\beta}\\ =& 2g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,0) - 2g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t)\\ =& R(x,0) - 2g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t),\end{aligned} \tag{53}$$

$$-2g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t) = -R(x,0) + \Delta_0 F(x,t). \tag{54}$$

Combining (51) and (54), and using Lemma 6.3 we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \varphi(x)dV_t \geq& -\int_M R(x,0)\varphi(x)dV_0\\ &+\int_M \varphi(x)\Delta_0 F(x,t)\cdot dV_0\\ \geq& -\frac{C_8}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) + \int_M \varphi(x)\Delta_0 F(x,t)\cdot dV_0,\\ & 0\leq t\leq T.\end{aligned} \tag{55}$$

Lemma 6.4. *For any $t\in[0,T]$ we always have*

$$\int_M \varphi(x)\Delta_0 F(x,t)\cdot dV_0 = \int_M F(x,t)\Delta_0\varphi(x)\cdot dV_0. \tag{56}$$

Proof. By Assumption C and Lemma 2.3 we know that for any integers $m\geq 0$, we have

$$\sup_{x\in M}|\nabla^m R_{ijkl}(x,t)|^2 \leq C(n,m)\left[\Theta\left(\frac{1}{t}\right)^m + \Theta^{\frac{m}{2}+1}\right],\ 0\leq t\leq T, \tag{57}$$

which implies

$$\sup_{x\in M}|\nabla^t R(x,t)|^2 \leq C(n)\left[\frac{\Theta}{t} + \Theta^{\frac{3}{2}}\right],\ 0\leq t\leq T, \tag{58}$$

where ∇^t denote the covariant derivatives with respect to ds_t^2. (58) can be written as

$$g^{ij}(x,t)\frac{\partial R(x,t)}{\partial x^i}\cdot\frac{\partial R(x,t)}{\partial x^j}\leq C(n)\left[\frac{\Theta}{t}+\Theta^{\frac{3}{2}}\right],\quad \text{on } M\times[0,T]. \tag{59}$$

From (49) it follows that $g^{ij}(x,0)\leq g^{ij}(x,t)$ on $M\times[0,T]$, so that, in consequence of (59),

$$g^{ij}(x,0)\frac{\partial R(x,t)}{\partial x^i}\cdot\frac{\partial R(x,t)}{\partial x^j}\leq C(n)\left[\frac{\Theta}{t}+\Theta^{\frac{3}{2}}\right],\quad \text{on } M\times[0,T],$$

$$\sup_{x\in M}|\widetilde{\nabla}R(x,t)|_0^2\leq C(n)\left[\frac{\Theta}{t}+\Theta^{\frac{3}{2}}\right],\ 0\leq t\leq T. \tag{60}$$

On the other hand, from (11) we know that

$$\frac{\partial}{\partial t}\widetilde{\nabla}_iF(x,t)=\widetilde{\nabla}_i\left[\frac{\partial}{\partial t}F(x,t)\right]=-\widetilde{\nabla}_iR(x,t), \tag{61}$$

which together with (14) and (60) implies

$$\begin{aligned}|\widetilde{\nabla}F(x,t)|_0&\leq\int_0^t|\widetilde{\nabla}R(x,s)|_0ds\\&\leq\int_0^t\sqrt{C(n)}\left[\frac{\Theta}{s}+\Theta^{\frac{3}{2}}\right]^{\frac{1}{2}}ds\\&\leq\widetilde{C}(n)\left[\sqrt{\Theta t}+\Theta^{\frac{3}{4}}t\right],\ \forall\ x\in M,\ 0\leq t\leq T.\end{aligned} \tag{62}$$

Combining (33), (36), (41) and (62) shows that for any fixed $t\in[0,T]$ we can integrate by part:

$$\begin{aligned}\int_M\varphi(x)\Delta_0F(x,t)\cdot dV_0&=\int_M\varphi(x)\cdot g^{ij}(x,0)\widetilde{\nabla}_i\widetilde{\nabla}_jF(x,t)\cdot dV_0\\&=-\int_M g^{ij}(x,0)\cdot\widetilde{\nabla}_i\varphi(x)\cdot\widetilde{\nabla}_jF(x,t)\cdot dV_0\\&=\int_M F(x,t)\cdot g^{ij}(x,0)\widetilde{\nabla}_j\widetilde{\nabla}_i\varphi(x)\cdot dV_0\\&=\int_M F(x,t)\Delta_0\varphi(x)\cdot dV_0.\end{aligned}$$

Thus the Lemma is true. q.e.d.

Combining (55) and Lemma 6.4 yields

$$\frac{\partial}{\partial t}\int_M \varphi(x)dV_t \geq -\frac{C_8}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) + \int_M F(x,t)\Delta_0\varphi(x)\cdot dV_0, \quad 0\leq t\leq T, \tag{63}$$

which together with (15) and (41) implies

$$\frac{\partial}{\partial t}\int_M \varphi(x)dV_t \geq -\frac{C_8}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) + \frac{C_6}{a^2}\int_M F(x,t)\varphi(x)dV_0, \quad 0\leq t\leq T, \tag{64}$$

$$-\frac{\partial}{\partial t}\int_M \varphi(x)dV_t \leq \frac{C_8}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) - \frac{C_6}{a^2}\int_M F(x,t)\varphi(x)dV_0, \quad 0\leq t\leq T. \tag{65}$$

Integrating (65) from 0 to t gives

$$\int_M \varphi(x)dV_0 - \int_M \varphi(x)dV_t \leq \frac{C_8 t}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) - \frac{C_6}{a^2}\int_0^t\int_M F(x,s)\varphi(x)dV_0 ds. \tag{66}$$

Since $dV_t = e^{F(x,t)}dV_0$, from (66) it follows that

$$\int_M [1-e^{F(x,t)}]\varphi(x)dV_0 \leq \frac{C_8 t}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) - \frac{C_6}{a^2}\int_0^t\int_M F(x,s)\varphi(x)dV_0 ds, \quad 0\leq t\leq T. \tag{67}$$

By (13) and (14) we obtain

$$0\geq F(x,s)\geq F(x,t), \quad \forall\ x\in M,\ 0\leq s\leq t\leq T. \tag{68}$$

Combining (67) and (68) yields

$$\begin{aligned}
&\int_M [1-e^{F(x,t)}]\varphi(x)dV_0 \\
&\qquad \le \frac{C_8 t}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) \\
&\qquad\quad - \frac{C_6}{a^2}\int_0^t\int_M F(x,t)\varphi(x)dV_0 ds \\
&\qquad = \frac{C_8 t}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) \\
&\qquad\quad - \frac{C_6 t}{a^2}\int_M F(x,t)\varphi(x)dV_0, \quad 0 \le t \le T.
\end{aligned} \tag{69}$$

Now we define

$$F_{\min}(t) = \inf_{x\in M} F(x,t), \quad 0 \le t \le T. \tag{70}$$

Using (14), (36) and (68) we have

$$\begin{cases}
F_{\min}(0) = 0, \\
0 \ge F_{\min}(t) \ge -4n^2\sqrt{\Theta}t, & 0 \le t \le T, \\
F_{\min}(s) \ge F_{\min}(t), & 0 \le s \le t \le T.
\end{cases} \tag{71}$$

Combining (15) and Lemma 6.3 we get

$$\begin{aligned}
\int_M F(x,t)\varphi(x)dV_0 &\ge F_{\min}(t)\int_M \varphi(x)dV_0 \\
&\ge C_7 F_{\min}(t)\cdot \operatorname{Vol}(B_0(x_0,a)), \\
&\qquad\qquad 0 \le t \le T.
\end{aligned} \tag{72}$$

Substituting (72) into (69) yields

$$\begin{aligned}
&\int_M [1-e^{F(x,t)}]\varphi(x)dV_0 \\
&\quad \le \frac{C_8 t}{a^\theta}\operatorname{Vol}(B_0(x_0,a)) - \frac{C_6 C_7 t}{a^2}F_{\min}(t)\cdot\operatorname{Vol}(B_0(x_0,a)) \\
&\quad = \left[\frac{C_8 t}{a^\theta} - \frac{C_6 C_7 t}{a^2}F_{\min}(t)\right]\cdot \operatorname{Vol}(B_0(x_0,a)), \quad 0 \le t \le T.
\end{aligned} \tag{73}$$

It is easy to see that

$$1 - e^F \ge -\frac{F}{2}, \quad \text{for } 0 \ge F \ge -1, \tag{74}$$

which implies

$$\begin{aligned}\int_M [1-e^{F(x,t)}]\varphi(x)dV_0 &= \int_{\{F\geq -1\}}(1-e^F)\varphi(x)dV_0\\ &\quad+\int_{\{F<-1\}}(1-e^F)\varphi(x)dV_0\\ &\geq -\frac{1}{2}\int_{\{F\geq -1\}}F(x,t)\varphi(x)dV_0\\ &\quad+\frac{1}{2}\int_{\{F<-1\}}\varphi(x)dV_0.\end{aligned} \tag{75}$$

Since every term on the right-hand side of (75) is nonnegative, we have

$$-\int_{\{F\geq -1\}}F(x,t)\varphi(x)dV_0 \leq 2\int_M [1-e^{F(x,t)}]\varphi(x)dV_0, \tag{76}$$

$$\int_{\{F<-1\}}\varphi(x)dV_0 \leq 2\int_M [1-e^{F(x,t)}]\varphi(x)dV_0, \tag{77}$$

$$\begin{aligned}-\int_{\{F<-1\}}F(x,t)\varphi(x)dV_0 &\leq -F_{\min}(t)\int_{\{F<-1\}}\varphi(x)dV_0\\ &\leq -2F_{\min}(t)\int_M [1-e^{F(x,t)}]\varphi(x)dV_0.\end{aligned} \tag{78}$$

Combining (76) and (78) we get

$$\begin{aligned}&-\int_M F(x,t)\varphi(x)dV_0\\ &\qquad\leq 2[1-F_{\min}(t)]\int_M [1-e^{F(x,t)}]\varphi(x)dV_0,\end{aligned} \tag{79}$$

which together with (73) implies

$$\begin{aligned}&-\int_M F(x,t)\varphi(x)dV_0\\ &\quad\leq 2[1-F_{\min}(t)]\left[\frac{C_8 t}{a^\theta}-\frac{C_6C_7 t}{a^2}F_{\min}(t)\right]\cdot \mathrm{Vol}(B_0(x_0,a)),\\ &\qquad\qquad 0\leq t\leq T.\end{aligned} \tag{80}$$

On the other hand, from (36) and (41) it follows that

$$
\begin{aligned}
-\int_M & F(x,t)\varphi(x)dV_0 \\
&\geq -\int_{B_0(x_0,a)} F(x,t)\varphi(x)dV_0 \\
&\geq -\int_{B_0(x_0,a)} F(x,t)e^{-C_6[1+\frac{1}{a}\gamma_0(x,x_0)]}dV_0 \qquad (81)\\
&\geq -\int_{B_0(x_0,a)} F(x,t)e^{-C_6[1+\frac{a}{a}]}dV_0 \\
&= -e^{-2C_6}\int_{B_0(x_0,a)} F(x,t)dV_0.
\end{aligned}
$$

Combining (80) and (81) yields

Lemma 6.5. *For any fixed point $x_0 \in M$ and constant $1 \leq a < +\infty$, we have*

$$
\begin{aligned}
-\int_{B_0(x_0,a)} F(x,t)dV_0 \leq & 2e^{2C_6}[1-F_{\min}(t)] \\
& \cdot\left[\frac{C_8 t}{a^\theta} - \frac{C_6 C_7 t}{a^2}F_{\min}(t)\right]\cdot \mathrm{Vol}(B_0(x_0,a)), \qquad (82)\\
& 0 \leq t \leq T.
\end{aligned}
$$

The next step is to estimate $F_{\min}(t)$ in terms of $\int_{B_0(x_0,a)} F(x,t)dV_0$. To do this we need to use the Green's function on M. Suppose $G_0(x,y)$ is the Green's function on M with respect to the metric $g_{ij}(x,0)$:

$$
\begin{cases} G_0(x,y) > 0, \\ \\ \Delta_0 G_0(x,y) = -\delta_x(y), \end{cases} \qquad x,\ y \in M, \qquad (83)
$$

where $\delta_x(y)$ denotes the Delta function.

Lemma 6.6. *There exist constants $0 < C_9, C_{10}, C_{11} < +\infty$ depending only on n and C_2 such that for $\forall\ x, y \in M$,*

(84)

$$
\frac{C_9\gamma_0(x,y)^2}{\mathrm{Vol}(B_0(x,\gamma_0(x,y)))} \leq G_0(x,y) \leq \frac{C_{10}\gamma_0(x,y)^2}{\mathrm{Vol}(B_0(x,\gamma_0(x,y)))},
$$

$$
|\widetilde{\nabla} G_0(x,y)|_0 \leq \frac{C_{11}\gamma_0(x,y)}{\mathrm{Vol}(B_0(x,\gamma_0(x,y)))}. \qquad (85)
$$

Proof. Using the result of P. Li and S.T. Yau [30], we know that there exist two constants $0 < C_{12}, C_{13} < +\infty$ depending only on n such that

$$\begin{aligned} C_{12}\int_{\gamma_0(x,y)^2}^{+\infty}\frac{dt}{\operatorname{Vol}(B_0(x,\sqrt{t}))} &\le G_0(x,y) \\ &\le\ C_{13}\int_{\gamma_0(x,y)^2}^{+\infty}\frac{dt}{\operatorname{Vol}(B_0(x,\sqrt{t}))}, \\ &\qquad \forall\ x,y\in M. \end{aligned} \tag{86}$$

By (29) in Assumption C we obtain

$$\begin{aligned} &\frac{C_2t^2}{\gamma_0(x,y)^4}\operatorname{Vol}(B_0(x,\gamma_0(x,y)))\le \operatorname{Vol}(B_0(x,\sqrt{t})) \\ &\le\frac{t^n}{\gamma_0(x,y)^{2n}}\operatorname{Vol}(B_0(x,\gamma_0(x,y))),\quad \text{for } t\ge\gamma_0(x,y)^2, \end{aligned} \tag{87}$$

which, together with (86) implies that (84) is true. Thus (85) follows from the S.Y. Cheng and S.T. Yau [12] gradient estimate for harmonic functions. q.e.d.

Now we fix a point $x_0 \in M$. For any constant $\alpha > 0$ we define

$$\Omega_\alpha = \{y\in M | G_0(x_0,y)>\alpha\}. \tag{88}$$

Since (29) in Assumption C implies

$$\begin{aligned} C_2\gamma^4\cdot\operatorname{Vol}(B_0(x,1))\le\operatorname{Vol}(B_0(x,\gamma))\le\gamma^{2n}\cdot\operatorname{Vol}(B_0(x,1)), \\ \forall\ x\in M,\ 1\le\gamma<+\infty; \end{aligned} \tag{89}$$

thus by (84) we know that $G_0(x,y)$ satisfies

$$\begin{aligned} \frac{C_9}{\gamma_0(x,y)^{2n-2}\cdot\operatorname{Vol}(B_0(x,1))} &\le G_0(x,y) \\ &\le\ \frac{C_{10}}{\gamma_0(x,y)^2\cdot C_2\operatorname{Vol}(B_0(x,1))}, \\ &\qquad \forall\ x,\ y\in M,\ \gamma_0(x,y)\ge 1. \end{aligned} \tag{90}$$

From (90) it follows that $G_0(x,y)$ do exist and decay to zero as $\gamma_0(x,y)\to+\infty$. Thus for any constant $\alpha>0$, $\overline{\Omega}_\alpha\subset M$ is a compact subset of M and

$$\partial\Omega_\alpha=\{y\in M|G_0(x_0,y)=\alpha\}. \tag{91}$$

Since $G_0(x_0,y)-\alpha\equiv 0$ on $\partial\Omega_\alpha$, by (83) it is easy to see that for any function $\mathcal{U}(y)\in C^2(\overline{\Omega}_\alpha)$ we have

$$\begin{aligned}\mathcal{U}(x_0)=&\int_{\Omega_\alpha}[\alpha-G_0(x_0,y)]\Delta_0\mathcal{U}(y)dV_0(y)\\&-\int_{\partial\Omega_\alpha}\mathcal{U}(y)\frac{\partial G_0(x_0,y)}{\partial\nu}\,d\sigma(y),\end{aligned} \tag{92}$$

where $\vec{\nu}$ denotes the outer unit normal vectors of $\partial\Omega_\alpha$, $d\sigma(y)$ denotes the volume element of $\partial\Omega_\alpha$ at y with respect to the metric $g_{ij}(x,0)$.

For any fixed $t\in[0,T]$, let $\mathcal{U}(y)=F(y,t)$. Then from (92) we get

$$\begin{aligned}F(x_0,t)=&\int_{\Omega_\alpha}[\alpha-G_0(x_0,y)]\Delta_0F(y,t)dV_0(y)\\&-\int_{\partial\Omega_\alpha}F(y,t)\frac{\partial G_0(x_0,y)}{\partial\nu}\,d\sigma(y).\end{aligned} \tag{93}$$

Lemma 6.7. *We have*

$$\Delta_0F(y,t)\le R(y,0),\quad\forall\ y\in M,\ 0\le t\le T. \tag{94}$$

Proof. The use of (7) yields

$$g^{\alpha\overline{\beta}}(x,0)R_{\alpha\overline{\beta}}(x,t)\ge 0,\quad\text{on } M\times[0,T], \tag{95}$$

which together with (53) implies the lemma. q.e.d.

By the definition of Ω_α we have

$$\alpha-G_0(x_0,y)\le 0,\quad\forall\ y\in\Omega_\alpha; \tag{96}$$

thus from Lemma 6.7 and the facts that $R(y,0)\ge 0$ and $\alpha>0$,

$$\begin{aligned}[\alpha-G_0(x_0,y)]\Delta_0F(y,t)\ge&[\alpha-G_0(x_0,y)]R(y,0)\\\ge&-G_0(x_0,y)R(y,0),\ \forall\ y\in\Omega_\alpha.\end{aligned} \tag{97}$$

Substituting (97) into (93) gives

$$\begin{aligned} F(x_0,t) \geq & -\int_{\Omega_\alpha} G_0(x_0,y)R(y,0)dV_0(y) \\ & -\int_{\partial\Omega_\alpha} F(y,t)\ \frac{\partial G_0(x_0,y)}{\partial \nu}\ d\sigma(y). \end{aligned} \tag{98}$$

On the other hand, from (89) it follows that

$$\begin{aligned} \frac{1}{\gamma^{2n-2}\cdot \mathrm{Vol}(B_0(x_0,1))} \leq & \frac{\gamma^2}{\mathrm{Vol}(B_0(x_0,\gamma))} \\ \leq & \frac{1}{C_2\gamma^2\cdot \mathrm{Vol}(B_0(x_0,1))}, \\ & \text{for } 1\leq \gamma < +\infty. \end{aligned} \tag{99}$$

Now we assume that α satisfies

$$0<\alpha \leq \frac{1}{\mathrm{Vol}(B_0(x_0,1))}. \tag{100}$$

Then from (99) we know that there exists a number $\gamma(\alpha)\geq 1$ such that

$$\frac{\gamma(\alpha)^2}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}=\alpha. \tag{101}$$

For any $y\in\partial\Omega_\alpha$, from (91) it follows that $G_0(x_0,y)=\alpha$. Thus combining (84) and (101) we get: for $\forall\ y\in\partial\Omega_\alpha$,

$$\begin{aligned} \frac{C_9\gamma_0(x_0,y)^2}{\mathrm{Vol}(B_0(x_0,\gamma_0(x_0,y)))} \leq & \frac{\gamma(\alpha)^2}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))} \\ \leq & \frac{C_{10}\gamma_0(x_0,y)^2}{\mathrm{Vol}(B_0(x_0,\gamma_0(x_0,y)))}, \end{aligned} \tag{102}$$

$$\begin{aligned} C_9\frac{\gamma_0(x_0,y)^2}{\gamma(\alpha)^2} \leq & \frac{\mathrm{Vol}(B_0(x_0,\gamma_0(x_0,y)))}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))} \\ \leq & C_{10}\frac{\gamma_0(x_0,y)^2}{\gamma(\alpha)^2}, \end{aligned} \tag{103}$$

which together with (29) in Assumption C implies

$$C_{12}\gamma(\alpha)\leq \gamma_0(x_0,y)\leq C_{13}\gamma(\alpha), \quad \forall\ y\in\partial\Omega_\alpha, \tag{104}$$

where $0<C_{12},C_{13}<+\infty$ are constants depending only on n and C_2. Thus

$$B_0(x_0,C_{12}\gamma(\alpha))\subset\Omega_\alpha\subset B_0(x_0,C_{13}\gamma(\alpha)). \tag{105}$$

Combining (85), (102) and (104) yields

$$|\widetilde{\nabla} G_0(x_0,y)|_0 \le \frac{C_{14}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}, \quad \forall\ y \in \partial\Omega_\alpha, \tag{106}$$

$$\left|\frac{\partial G_0(x_0,y)}{\partial \nu}\right| \le \frac{C_{14}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}, \quad \forall\ y \in \partial\Omega_\alpha, \tag{107}$$

where $0 < C_{14} < +\infty$ is a constant depending only on n and C_2, and $\vec{\nu}$ is the outer unit normal vector of $\partial\Omega_\alpha$.

Since $F(y,t) \le 0$ on $M \times [0,T]$, we have, in consequence of (107),

$$\begin{aligned} &-\int_{\partial\Omega_\alpha} F(y,t)\frac{\partial G_0(x_0,y)}{\partial\nu}\, d\sigma(y) \\ &\ge \frac{C_{14}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{\partial\Omega_\alpha} F(y,t)d\sigma(y). \end{aligned} \tag{108}$$

Since $G_0(x_0,y) > 0,\ R(y,0) \ge 0$, from (105) it follows that

$$\begin{aligned} &\int_{\Omega_\alpha} G_0(x_0,y)R(y,0)dV_0(y) \\ &\le \int_{B_0(x_0,C_{13}\gamma(\alpha))} G_0(x_0,y)R(y,0)dV_0(y) \\ &\le \int_{B_0(x_0,1)} G_0(x_0,y)R(y,0)dV_0(y) \\ &\quad + \sum_{k=1}^{s}\int_{B_0(x_0,2^k)\setminus B_0(x_0,2^{k-1})} G_0(x_0,y)R(y,0)dV_0(y), \end{aligned} \tag{109}$$

where

$$s = 1 + \max\left\{\left[\frac{\log(C_{13}\gamma(\alpha))}{\log 2}\right],\ 0\right\}. \tag{110}$$

By (28) in Assumption C we get

$$0 \le R(y,0) \le 4n^2k_0, \quad \forall\ y \in M. \tag{111}$$

Thus

$$\begin{aligned} &\int_{B_0(x_0,1)} G_0(x_0,y)R(y,0)dV_0(y) \\ &\le 4n^2k_0\int_{B_0(x_0,1)} G_0(x_0,y)dV_0(y). \end{aligned} \tag{112}$$

Combining (29) of Assumption C, (84) and (112) we get

$$
\begin{aligned}
&\int_{B_0(x_0,1)} G_0(x_0,y)R(y,0)dV_0(y) \\
&\le 4n^2k_0 \int_{B_0(x_0,1)} \frac{C_{10}\gamma_0(x_0,y)^2}{\mathrm{Vol}(B_0(x_0,\gamma_0(x_0,y)))}\, dV_0(y) \\
&= 4n^2k_0C_{10}\sum_{k=1}^{\infty}\int_{B_0(x_0,\frac{1}{2^{k-1}})\setminus B_0(x_0,\frac{1}{2^k})} \frac{\gamma_0(x_0,y)^2 dV_0(y)}{\mathrm{Vol}(B_0(x_0,\gamma_0(x_0,y)))} \\
&\le 4n^2k_0C_{10}\sum_{k=1}^{\infty}\int_{B_0(x_0,\frac{1}{2^{k-1}})\setminus B_0(x_0,\frac{1}{2^k})} \frac{(\frac{1}{2})^{2k-2}}{\mathrm{Vol}(B_0(x_0,\frac{1}{2^k}))}\, dV_0(y) \\
&\le 4n^2k_0C_{10}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k-2} \frac{\mathrm{Vol}(B_0(x_0,\frac{1}{2^{k-1}}))}{\mathrm{Vol}(B_0(x_0,\frac{1}{2^k}))} \\
&\le 4n^2k_0C_{10}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k-2}\cdot 2^{2n} \le C_{15},
\end{aligned} \tag{113}
$$

where $0 < C_{15} < +\infty$ is a constant depending only on n, k_0 and C_2. On the other hand, from (84) it follows that

$$
\begin{aligned}
&\int_{B_0(x_0,2^k)\setminus B_0(x_0,2^{k-1})} G_0(x_0,y)R(y,0)dV_0(y) \\
&\le \int_{B_0(x_0,2^k)\setminus B_0(x_0,2^{k-1})} \frac{C_{10}\gamma_0(x_0,y)^2}{\mathrm{Vol}(B_0(x_0,\gamma_0(x_0,y)))}\, R(y,0)dV_0(y) \\
&\le \int_{B_0(x_0,2^k)\setminus B_0(x_0,2^{k-1})} \frac{2^{2k}C_{10}}{\mathrm{Vol}(B_0(x_0,2^{k-1}))} R(y,0)dV_0(y) \\
&\le \frac{4^kC_{10}}{\mathrm{Vol}(B_0(x_0,2^{k-1}))}\int_{B_0(x_0,2^k)} R(y,0)dV_0(y),
\end{aligned} \tag{114}
$$

which together with (29) and (30) of Assumption C yields

$$
\begin{aligned}
&\int_{B_0(x_0,2^k)\setminus B_0(x_0,2^{k-1})} G_0(x_0,y)R(y,0)dV_0(y) \\
&\le \frac{4^kC_{10}}{\mathrm{Vol}(B_0(x_0,2^{k-1}))}\cdot\frac{C_3}{(2^k+1)^{\theta}}\,\mathrm{Vol}(B_0(x_0,2^k)) \\
&\le C_3C_{10}(2^k)^{2-\theta}\frac{\mathrm{Vol}(B_0(x_0,2^k))}{\mathrm{Vol}(B_0(x_0,2^{k-1}))} \le 2^{2n}C_3C_{10}(2^k)^{2-\theta}.
\end{aligned} \tag{115}
$$

Combining (109), (113) and (115) we get

$$\begin{aligned}&\int_{\Omega_\alpha} G_0(x_0,y)R(y,0)dV_0(y)\\ &\leq C_{15}+\sum_{k=1}^{s} 2^{2n}C_3C_{10}(2^k)^{2-\theta}\leq C_{16}(2^s)^{2-\theta},\end{aligned} \tag{116}$$

where $0<C_{16}<+\infty$ is a constant depending only on n, k_0,θ,C_2 and C_3. Combining (110) and (116) implies

$$\int_{\Omega_\alpha} G_0(x_0,y)R(y,0)dV_0(y)\leq C_{17}\gamma(\alpha)^{2-\theta}, \tag{117}$$

where $0<C_{17}<+\infty$ is a constant depending only on n,k_0,θ,C_2 and C_3. Combining (98), (108) and (117) yields

$$\begin{aligned}F(x_0,t)\geq &-C_{17}\gamma(\alpha)^{2-\theta}\\ &+\frac{C_{14}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{\partial\Omega_\alpha}F(y,t)d\sigma(y).\end{aligned} \tag{118}$$

Suppose $\alpha>0$ satisfies (100). Then for any $\beta\in\left[\frac{\alpha}{2},\alpha\right]$, from (118) it follows that

$$\begin{aligned}F(x_0,t)\geq &-C_{17}\gamma(\beta)^{2-\theta}\\ &+\frac{C_{14}\gamma(\beta)}{\mathrm{Vol}(B_0(x_0,\gamma(\beta)))}\int_{\partial\Omega_\beta}F(y,t)d\sigma(y),\\ &\frac{\alpha}{2}\leq\beta\leq\alpha.\end{aligned} \tag{119}$$

By the definition of $\gamma(\beta)$ in (101),

$$\begin{aligned}\frac{\gamma(\alpha)^2}{2\cdot\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}&\leq\frac{\gamma(\beta)^2}{\mathrm{Vol}(B_0(x_0,\gamma(\beta)))}\\ &\leq\frac{\gamma(\alpha)^2}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))},\quad \frac{\alpha}{2}\leq\beta\leq\alpha,\\ \frac{\gamma(\beta)^2}{\gamma(\alpha)^2}&\leq\frac{\mathrm{Vol}(B_0(x_0,\gamma(\beta)))}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\leq 2\,\frac{\gamma(\beta)^2}{\gamma(\alpha)^2},\ \frac{\alpha}{2}\leq\beta\leq\alpha,\end{aligned} \tag{120}$$

which together with (29) of Assumption C implies

$$C_{18}\gamma(\alpha)\leq\gamma(\beta)\leq C_{19}\gamma(\alpha),\ \frac{\alpha}{2}\leq\beta\leq\alpha, \tag{121}$$

where $0 < C_{18}, C_{19} < +\infty$ are constants depending only on n and C_2. Combining (119), (120) and (121) gives

$$\begin{aligned} F(x_0,t) \geq & - C_{20}\gamma(\alpha)^{2-\theta} \\ & + \frac{C_{20}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{\partial\Omega_\beta} F(y,t)d\sigma(y), \\ & \frac{\alpha}{2} \leq \beta \leq \alpha, \end{aligned} \tag{122}$$

where $0 < C_{20} < +\infty$ is a constant depending only on n, k_0, θ, C_2 and C_3. Integrating (122) from $\frac{\alpha}{2}$ to α, we obtain

$$\begin{aligned} F(x_0,t) = & \frac{2}{\alpha}\int_{\frac{\alpha}{2}}^{\alpha} F(x_0,t)d\beta \\ \geq & - C_{20}\gamma(\alpha)^{2-\theta} \\ & + \frac{C_{20}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\cdot\frac{2}{\alpha}\int_{\frac{\alpha}{2}}^{\alpha}\int_{\partial\Omega_\beta} F(y,t)d\sigma(y)d\beta. \end{aligned} \tag{123}$$

For any $y \in \partial\Omega_\beta$, if we use $\vec{\nu}$ to denote the outer unit normal vectors of $\partial\Omega_\beta$, then

$$d\beta = \frac{\partial G_0(x_0,y)}{\partial\nu}\, d\nu. \tag{124}$$

Combining (107), (120), (121) and (124) we know that

$$\begin{aligned} d\sigma(y)d\beta &= \frac{\partial G_0(x_0,y)}{\partial\nu}\, d\sigma(y)d\nu \\ &= \left|\frac{\partial G_0(x_0,y)}{\partial\nu}\right| d\sigma(y)|d\nu| = \left|\frac{\partial G_0(x_0,y)}{\partial\nu}\right| dV_0(y) \\ &\leq \frac{C_{14}\gamma(\beta)}{\mathrm{Vol}(B_0(x_0,\gamma(\beta)))}\, dV_0(y) \\ &\leq \frac{C_{21}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\, dV_0(y), \quad y \in \partial\Omega_\beta,\ \frac{\alpha}{2} \leq \beta \leq \alpha, \end{aligned} \tag{125}$$

where $0 < C_{21} < +\infty$ is a constant depending only on n and C_2. Since

$F(y,t)\le 0$ on $M\times[0,T]$, from (105) and (125) it follows that

$$
\begin{aligned}
&\int_{\frac{\alpha}{2}}^{\alpha}\int_{\partial\Omega_\beta} F(y,t)d\sigma(y)d\beta \\
&\qquad\ge \frac{C_{21}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{\Omega_{\frac{\alpha}{2}}\setminus\Omega_\alpha} F(y,t)dV_0(y) \\
&\qquad\ge \frac{C_{21}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{\Omega_{\frac{\alpha}{2}}} F(y,t)dV_0(y) \\
&\qquad\ge \frac{C_{21}\gamma(\alpha)}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{B_0(x_0,C_{13}\gamma(\frac{\alpha}{2}))} F(y,t)dV_0(y).
\end{aligned} \tag{126}
$$

Combining (101) and (126) we obtain

$$
\begin{aligned}
&\frac{2}{\alpha}\int_{\frac{\alpha}{2}}^{\alpha}\int_{\partial\Omega_\beta} F(y,t)d\sigma(y)d\beta \\
&\qquad\ge \frac{2C_{21}}{\gamma(\alpha)}\int_{B_0(x_0,C_{13}\gamma(\frac{\alpha}{2}))} F(y,t)dV_0(y),
\end{aligned} \tag{127}
$$

which together with (123) implies

$$
\begin{aligned}
F(x_0,t)\ge &-C_{20}\gamma(\alpha)^{2-\theta} \\
&+\frac{2C_{20}C_{21}}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{B_0(x_0,C_{13}\gamma(\frac{\alpha}{2}))} F(y,t)dV_0(y).
\end{aligned} \tag{128}
$$

Given number a such that

$$
a\ge 1+\left(\sqrt{\frac{2}{C_2}}+1\right)C_{13}, \tag{129}
$$

we let

$$
\alpha = 2\left(\frac{a}{C_{13}}\right)^2 \frac{1}{\mathrm{Vol}(B_0(x_0,\frac{a}{C_{13}}))}. \tag{130}
$$

From (29) of Assumption C it follows that

$$
\begin{aligned}
0<\alpha \le& \frac{2}{C_2}\left(\frac{C_{13}}{a}\right)^2\frac{1}{\mathrm{Vol}(B_0(x_0,1))} \\
\le& \frac{1}{\mathrm{Vol}(B_0(x_0,1))};
\end{aligned} \tag{131}
$$

thus α satisfies (100). Combining (101) and (130) shows that $\gamma(\frac{\alpha}{2})$ can be chosen as

$$\gamma(\frac{\alpha}{2}) = \frac{a}{C_{13}}. \tag{132}$$

Now (128) implies

$$\begin{aligned} F(x_0,t) \geq & - C_{20}\gamma(\alpha)^{2-\theta} \\ & + \frac{2C_{20}C_{21}}{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}\int_{B_0(x_0,a)} F(y,t)dV_0(y). \end{aligned} \tag{133}$$

By (121) we get

$$C_{18}\gamma(\alpha) \leq \gamma(\frac{\alpha}{2}) \leq C_{19}\gamma(\alpha), \tag{134}$$

which together with (132) yields

$$\frac{a}{C_{13}C_{19}} \leq \gamma(\alpha) \leq \frac{a}{C_{13}C_{18}}. \tag{135}$$

Combining (29) of Assumption C and (135) we have

$$C_{22} \leq \frac{\mathrm{Vol}(B_0(x_0,\gamma(\alpha)))}{\mathrm{Vol}(B_0(x_0,a))} \leq C_{23}, \tag{136}$$

where $0 < C_{22}, C_{23} < +\infty$ are constants depending only on n and C_2. Combining (133), (135) and (136) shows that the following lemma is true:

Lemma 6.8. *For any fixed point $x_0 \in M$ and number a which satisfies (129), we have*

$$\begin{aligned} F(x_0,t) \geq & - C_{24}a^{2-\theta} \\ & + \frac{C_{24}}{\mathrm{Vol}(B_0(x_0,a))}\int_{B_0(x_0,a)} F(y,t)dV_0(y), \\ & \qquad 0 \leq t \leq T, \end{aligned} \tag{137}$$

where $0 < C_{24} < +\infty$ is a constant depending only on n, k_0, θ, C_2 and C_3.

For any fixed $t \in [0,T]$, we choose a point $x_0 \in M$ such that $F(x_0,t) \leq \frac{1}{2}F_{\min}(t) \leq 0$. Suppose the number a satisfies (129). Then from Lemma 6.5 and Lemma 6.8 it follows that

$$
\begin{aligned}
F_{\min}(t) \geq 2F(x_0,t) \geq & -2C_{24}a^{2-\theta} \\
& + \frac{2C_{24}}{\mathrm{Vol}(B_0(x_0,a))}\int_{B_0(x_0,a)} F(y,t)dV_0(y) \\
\geq & -2C_{24}a^{2-\theta} - 4C_{24}e^{2C_6}[1-F_{\min}(t)] \\
& \cdot \left[\frac{C_8 t}{a^\theta} - \frac{C_6C_7t}{a^2}F_{\min}(t)\right],\ 0 \leq t \leq T.
\end{aligned} \tag{138}
$$

Now we let

$$
\begin{aligned}
a = & 1 + \left(\sqrt{\frac{2}{C_2}}+1\right)C_{13} \\
& + 4e^{C_6}\sqrt{C_6C_7C_{24}}(t+2)^{\frac{1}{2}}[1-F_{\min}(t)]^{\frac{1}{2}}.
\end{aligned} \tag{139}
$$

Then a satisfies (129). Substituting (139) into (138), gives

$$
\begin{aligned}
F_{\min}(t) \geq & -C_{25}(t+2)^{\frac{2-\theta}{2}}[1-F_{\min}(t)]^{\frac{2-\theta}{2}} \\
& + \frac{1}{4}F_{\min}(t), \quad 0 \leq t \leq T,
\end{aligned} \tag{140}
$$

where $0 < C_{25} < +\infty$ is a constant depending only on n, k_0, θ, C_2 and C_3. From (140) we get

$$
\begin{aligned}
1 - F_{\min}(t) \leq \left(1+\frac{4}{3}C_{25}\right)(t+2)^{\frac{2-\theta}{2}}[1-F_{\min}(t)]^{\frac{2-\theta}{2}}, \\
0 \leq t \leq T,
\end{aligned} \tag{141}
$$

$$
1 - F_{\min}(t) \leq \left(1+\frac{4}{3}C_{25}\right)^{\frac{2}{\theta}}(t+2)^{\frac{2-\theta}{\theta}}, \quad 0 \leq t \leq T. \tag{142}
$$

Thus

$$
F_{\min}(t) \geq -C_{26}(t+2)^{\frac{2-\theta}{\theta}}, \quad 0 \leq t \leq T, \tag{143}
$$

where $0 < C_{26} < +\infty$ is a constant depending only on n, k_0, θ, C_2 and C_3. Combining (70) and (143) yields

$$
F(x,t) \geq -C_{26}(t+2)^{\frac{2-\theta}{\theta}},\ \text{on } M \times [0,T]. \tag{144}
$$

Thus the proof of Theorem 6.2 is completed. Since the constant C_2 in (23) depends only on n, as we already mentioned, Theorem 6.1 now follows from Theorem 6.2.

Remark. If the constant $\theta = 2$ in Assumption B, then the corresponding statement of Theorem 6.1 is

$$F(x,t) \geq -C(n,k_0,C_1)\cdot \log(t+2), \qquad \text{on } M\times[0,T], \tag{145}$$

where $0 < C(n,k_0,C_1) < +\infty$ is a constant depending only on n, k_0 and C_1, and is independent of Θ_0 and T. The proof of (145) is the same as the proof of Theorem 6.1. Under the assumptions of Theorem 1.1, (145) was proved by the author of this paper in [43] in 1990.

Corollary 6.9. *Under Assumption B, there exists a constant* $0 < C_{27} < +\infty$ *depending only on* n, k_0, θ *and* C_1 *such that*

$$g_{\alpha\overline{\beta}}(x,0) \geq g_{\alpha\overline{\beta}}(x,t) \geq e^{-C_{27}(t+2)^{\frac{2-\theta}{\theta}}}\cdot g_{\alpha\overline{\beta}}(x,0), \qquad \text{on } M\times[0,T], \tag{146}$$

$$ds_0^2 \geq ds_t^2 \geq e^{-C_{27}(t+2)^{\frac{2-\theta}{\theta}}}\cdot ds_0^2, \qquad 0\leq t\leq T. \tag{147}$$

Proof. Under Assumption B, from Theorem 6.1 we know that there exists a constant $0 < C_{27} < +\infty$ depending only on n, k_0, θ and C_1 such that

$$F(x,t) \geq -C_{27}(t+2)^{\frac{2-\theta}{\theta}}, \quad \text{on } M\times[0,T]. \tag{148}$$

Combining (9) and (148) we have

$$\frac{\det(g_{\alpha\overline{\beta}}(x,t))}{\det(g_{\alpha\overline{\beta}}(x,0))} \geq e^{-C_{27}(t+2)^{\frac{2-\theta}{\theta}}}, \qquad \text{on } M\times[0,T], \tag{149}$$

which together with (49) implies (146) and (147). q.e.d.

7. Long time existence

In this section, we are going to prove the long time existence for the solution to the Ricci flow equation

$$\begin{cases} \frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t), \\ g_{ij}(x,0) = \widetilde{g}_{ij}(x), \quad x\in M \end{cases} \tag{1}$$

under the following assumption:

Assumption D. $(M, \widetilde{g}_{ij}(x))$ is a complex n–dimensional complete noncompact Kähler manifold which satisfies

$$(2) \qquad \text{(i)} \quad 0 \le -R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,0) \le k_0, \quad x \in M,$$

$$(3) \qquad \text{(ii)} \quad \int_{B_0(x_0,\gamma)} R(x,0)dV_0 \le \frac{C_1}{(\gamma+1)^\theta} \cdot \mathrm{Vol}(B_0(x_0,\gamma)), \quad x_0 \in M, \ 0 \le \gamma < +\infty,$$

where $0 < \theta < 2$ and $0 < k_0, C_1 < +\infty$ are constants.

Under Assumption D, from Theorem 2.4 we know that the Ricci flow equation (1) has a smooth solution $g_{ij}(x,t) > 0$ for a short time:

$$0 \le t \le \frac{\theta_0(n)}{k_0} \tag{4}$$

and satisfies the following estimates:

$$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \le \frac{C(n,m)\cdot k_0^2}{t^m}, \ 0 \le t \le \frac{\theta_0(n)}{k_0}, \ m \ge 0, \tag{5}$$

where $0 < \theta_0(n) < +\infty$ is a constant depending only on n, $0 < C(n,m) < +\infty$ are constants depending only on n and m. Thus we have

Lemma 7.1. *Under Assumption D, there exists a constant* $0 < T < +\infty$ *such that the following Assumption E holds on* $M \times [0,T]$.

Assumption E. Suppose $(M, \widetilde{g}_{ij}(x))$ is a complex n–dimensional complete noncompact Kähler manifold, and $g_{ij}(x,t) > 0$ is a smooth solution to the Ricci flow equation (1) on $M \times [0,T]$ such that

$$(6) \qquad \text{(i)} \quad 0 \le -R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,0) \le k_0, \quad x \in M,$$

$$(7) \qquad \text{(ii)} \quad \int_{B_0(x_0,\gamma)} R(x,0)dV_0 \le \frac{C_1}{(\gamma+1)^\theta} \cdot \mathrm{Vol}(B_0(x_0,\gamma)), \quad x_0 \in M, \ 0 \le \gamma < +\infty,$$

$$(8) \qquad \text{(iii)} \quad \frac{\theta_0(n)}{k_0} \leq T < +\infty,$$

$$(9) \qquad \text{(iv)} \quad \sup_{x \in M} |\nabla^m R_{ijkl}(x,t)|^2 \leq \frac{C(n,m) \cdot k_0^2}{t^m},$$

$$0 \leq t \leq \frac{\theta_0(n)}{k_0}, \; m \geq 0,$$

$$(10) \qquad \text{(v)} \quad \sup_{M \times [0,T]} |R_{ijkl}(x,t)|^2 \leq \Theta,$$

where $0 < \Theta < +\infty$ is a constant, and the other constants in (6), (7), (8) and (9) are defined by (2), (3), (4) and (5).

Lemma 7.2. *Under Assumption E, $g_{ij}(x,t)$ are Kähler metrics for any $t \in [0,T]$ and satisfy the following estimates:*

$$(11) \qquad -R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) \geq 0, \quad \textit{on } M \times [0,T],$$

$$(12) \qquad F(x,t) \geq -C_2(t+2)^{\frac{2-\theta}{\theta}}, \quad \textit{on } M \times [0,T],$$

$$(13) \qquad g_{\alpha\overline{\beta}}(x,0) \geq g_{\alpha\overline{\beta}}(x,t) \geq e^{-C_2(t+2)^{\frac{2-\theta}{\theta}}} \cdot g_{\alpha\overline{\beta}}(x,0),$$

$$\textit{on } M \times [0,T],$$

where $0 < C_2 < +\infty$ is a constant depending only on n, k_0, θ and C_1.

Proof. Since Assumption E implies Assumption B in §6, by Theorem 5.3, $g_{ij}(x,t)$ are Kähler metrics for any $t \in [0,T]$, by (6) of §6, (11) is true, by Theorem 6.1 and Corollary 6.9, (12) and (13) are true. q.e.d.

To prove the long time existence for the solution to the Ricci flow equation (1) we have to establish some prior estimates of $g_{ij}(x,t)$ on $M \times [0,T]$ under Assumption E. More precisely, we are going to estimate the derivatives of $g_{ij}(x,t)$ only in terms of n, k_0, θ, C_1 and t. Especially they are independent of Θ. Since the Ricci flow equation (1) is the parabolic version of the complex Monge–Ampère equation on the Kähler manifolds, we know that inequality (13) is the parabolic version of the corresponding second order estimate for the Monge–Ampère equation. The derivative estimate for $g_{ij}(x,t)$ is the parabolic version of the corresponding third order estimate for the Monge–Ampère equation. The third order estimate for the Monge–Ampère equation was developed by E. Calabi in [8] and later used by S.T. Yau in [48]. In this section we want to establish the parabolic version of the third order estimate for the Monge–Ampère equation.

At the beginning we let

$$\tau_0 = \frac{\theta_0(n)}{k_0}. \tag{14}$$

We use $\widehat{\nabla} = \nabla^{\tau_0}$ to denote the covariant derivatives with respect to the metric $ds^2_{\tau_0}$, and $\widehat{\Delta} = \Delta_{\tau_0}$ to denote the Laplacian operator with respect to the metric $ds^2_{\tau_0}$. From (9) we get

$$\sup_{x\in M} |\widehat{\nabla}^m R_{ijkl}(x,\tau_0)|^2 \leq \widehat{C}(n,m) k_0^{m+2},\ m \geq 0, \tag{15}$$

where $0 < \widehat{C}(n,m) < +\infty$ are constants depending only on n and m. We also denote

$$\widehat{g}_{ij}(x) = g_{ij}(x,\tau_0), \quad x \in M, \tag{16}$$

$$\widehat{R}_{ijkl}(x) = R_{ijkl}(x,\tau_0), \quad x \in M. \tag{17}$$

Thus

$$\begin{aligned}
\frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x,t) =& -2R_{\alpha\overline{\beta}}(x,t) = 2\frac{\partial^2}{\partial z^\alpha \partial \overline{z}^\beta} \log\det(g_{\gamma\overline{\delta}}(x,t))\\
=&2\frac{\partial^2}{\partial z^\alpha \partial \overline{z}^\beta} \log \frac{\det(g_{\gamma\overline{\delta}}(x,t))}{\det(g_{\gamma\overline{\delta}}(x,\tau_0))} - 2R_{\alpha\overline{\beta}}(x,\tau_0)\\
=&2\widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\beta}} \log \frac{\det(g_{\gamma\overline{\delta}}(x,t))}{\det(g_{\gamma\overline{\delta}}(x,\tau_0))} - 2R_{\alpha\overline{\beta}}(x,\tau_0)\\
=&2\widehat{\nabla}_\alpha [g^{\gamma\overline{\delta}}(x,t)\widehat{\nabla}_{\overline{\beta}} g_{\gamma\overline{\delta}}(x,t)] - 2R_{\alpha\overline{\beta}}(x,\tau_0)\\
=&2g^{\gamma\overline{\delta}} \widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\beta}} g_{\gamma\overline{\delta}} + 2\widehat{\nabla}_\alpha g^{\gamma\overline{\delta}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\gamma\overline{\delta}} - 2\widehat{R}_{\alpha\overline{\beta}}.
\end{aligned} \tag{18}$$

On the other hand, since $g_{\alpha\overline{\beta}}(x,t)$ are Kähler metrics on M for any $t \in [0,T]$ by Lemma 7.2, we have

$$\begin{cases} \widehat{\nabla}_\alpha g_{\beta\overline{\gamma}} = \widehat{\nabla}_\beta g_{\alpha\overline{\gamma}}, \\ \\ \widehat{\nabla}_{\overline{\alpha}} g_{\beta\overline{\gamma}} = \widehat{\nabla}_{\overline{\gamma}} g_{\beta\overline{\alpha}}, \end{cases} \quad \text{on } M \times [0,T]. \tag{19}$$

Suppose we choose a coordinate system such that $\widehat{g}_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ at one point. We have the interchange formulas of the covariant derivatives:

Suppose $\{V_\alpha\}$ and $\{V_{\overline{\alpha}}\}$ are any covectors of $(1,0)$ type and $(0,1)$ type respectively. Then

$$\begin{cases} \widehat{\nabla}_\alpha \widehat{\nabla}_\beta V_\gamma = \widehat{\nabla}_\beta \widehat{\nabla}_\alpha V_\gamma, \\ \widehat{\nabla}_\alpha \widehat{\nabla}_\beta V_{\overline{\gamma}} = \widehat{\nabla}_\beta \widehat{\nabla}_\alpha V_{\overline{\gamma}}, \\ \widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\beta}} V_\gamma = \widehat{\nabla}_{\overline{\beta}} \widehat{\nabla}_\alpha V_\gamma + \widehat{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} V_\delta, \\ \widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\beta}} V_{\overline{\gamma}} = \widehat{\nabla}_{\overline{\beta}} \widehat{\nabla}_\alpha V_{\overline{\gamma}} - \widehat{R}_{\alpha\overline{\beta}\delta\overline{\gamma}} V_{\overline{\delta}}. \end{cases} \tag{20}$$

Using (19) and (20) we get

$$\begin{aligned} \widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\beta}} g_{\gamma\overline{\delta}} &= \widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\delta}} g_{\gamma\overline{\beta}} \\ &= \widehat{\nabla}_{\overline{\delta}} \widehat{\nabla}_\alpha g_{\gamma\overline{\beta}} + \widehat{R}_{\alpha\overline{\delta}\gamma\overline{\xi}} g_{\xi\overline{\beta}} - \widehat{R}_{\alpha\overline{\delta}\xi\overline{\beta}} g_{\gamma\overline{\xi}} \\ &= \widehat{\nabla}_{\overline{\delta}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + \widehat{R}_{\alpha\overline{\delta}\gamma\overline{\xi}} g_{\xi\overline{\beta}} - \widehat{R}_{\alpha\overline{\delta}\xi\overline{\beta}} g_{\gamma\overline{\xi}}, \end{aligned} \tag{21}$$

$$\widehat{\nabla}_{\overline{\delta}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} = \widehat{\nabla}_\gamma \widehat{\nabla}_{\overline{\delta}} g_{\alpha\overline{\beta}} + \widehat{R}_{\gamma\overline{\delta}\xi\overline{\beta}} g_{\alpha\overline{\xi}} - \widehat{R}_{\gamma\overline{\delta}\alpha\overline{\xi}} g_{\xi\overline{\beta}}. \tag{22}$$

Since $\widehat{R}_{\alpha\overline{\delta}\gamma\overline{\xi}} = \widehat{R}_{\gamma\overline{\delta}\alpha\overline{\xi}}$, from (21) and (22) it follows that

$$\widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\beta}} g_{\gamma\overline{\delta}} = \widehat{\nabla}_\gamma \widehat{\nabla}_{\overline{\delta}} g_{\alpha\overline{\beta}} + \widehat{R}_{\gamma\overline{\delta}\xi\overline{\beta}} g_{\alpha\overline{\xi}} - \widehat{R}_{\alpha\overline{\delta}\xi\overline{\beta}} g_{\gamma\overline{\xi}}. \tag{23}$$

Combining (21) and (23) implies

$$\begin{aligned} 2\widehat{\nabla}_\alpha \widehat{\nabla}_{\overline{\beta}} g_{\gamma\overline{\delta}} =& \widehat{\nabla}_{\overline{\delta}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + \widehat{\nabla}_\gamma \widehat{\nabla}_{\overline{\delta}} g_{\alpha\overline{\beta}} \\ &+ \widehat{R}_{\gamma\overline{\delta}\alpha\overline{\xi}} g_{\xi\overline{\beta}} + \widehat{R}_{\gamma\overline{\delta}\xi\overline{\beta}} g_{\alpha\overline{\xi}} - 2\widehat{R}_{\alpha\overline{\beta}\xi\overline{\delta}} g_{\gamma\overline{\xi}}, \end{aligned} \tag{24}$$

where we have also used the curvature property that $\widehat{R}_{\alpha\overline{\delta}\xi\overline{\beta}} = \widehat{R}_{\alpha\overline{\beta}\xi\overline{\delta}}$. On the other hand, it is easy to see that

$$\widehat{\nabla}_\alpha g^{\gamma\overline{\delta}} = -g^{\gamma\overline{\xi}} g^{\zeta\overline{\delta}} \widehat{\nabla}_\alpha g_{\zeta\overline{\xi}}. \tag{25}$$

Substituting (24) and (25) into (18) gives

$$\begin{aligned} \frac{\partial}{\partial t} g_{\alpha\overline{\beta}} =& g^{\gamma\overline{\delta}} \widehat{\nabla}_{\overline{\delta}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + g^{\gamma\overline{\delta}} \widehat{\nabla}_\gamma \widehat{\nabla}_{\overline{\delta}} g_{\alpha\overline{\beta}} + g^{\gamma\overline{\delta}} g_{\xi\overline{\beta}} \widehat{R}_{\gamma\overline{\delta}\alpha\overline{\xi}} \\ &+ g^{\gamma\overline{\delta}} g_{\alpha\overline{\xi}} \widehat{R}_{\gamma\overline{\delta}\xi\overline{\beta}} - 2\widehat{R}_{\alpha\overline{\beta}\xi\overline{\delta}} g^{\gamma\overline{\delta}} g_{\gamma\overline{\xi}} \\ &- 2g^{\gamma\overline{\xi}} g^{\zeta\overline{\delta}} \widehat{\nabla}_\alpha g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\gamma\overline{\delta}} - 2\widehat{R}_{\alpha\overline{\beta}}. \end{aligned} \tag{26}$$

Since by (121) of §5, $\widehat{R}_{\alpha\overline{\beta}\xi\overline{\delta}} g^{\gamma\overline{\delta}} g_{\gamma\overline{\xi}} = \widehat{R}_{\alpha\overline{\beta}\delta\overline{\delta}} = -\widehat{R}_{\alpha\overline{\beta}}$, where $\widehat{R}_{\alpha\overline{\beta}}$ denotes the Ricci curvature of $\widehat{g}_{\alpha\overline{\beta}}$, by (26) we get

$$\begin{aligned} \frac{\partial}{\partial t} g_{\alpha\overline{\beta}} =& g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\xi g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_\xi \widehat{\nabla}_{\overline{\zeta}} g_{\alpha\overline{\beta}} + g^{\sigma\overline{\eta}} g_{\xi\overline{\beta}} \widehat{R}_{\sigma\overline{\eta}\alpha\overline{\xi}} \\ &+ g^{\sigma\overline{\eta}} g_{\alpha\overline{\xi}} \widehat{R}_{\sigma\overline{\eta}\xi\overline{\beta}} - 2g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{\nabla}_\alpha g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}}. \end{aligned} \tag{27}$$

Since $g^{\alpha\overline{\beta}}g_{\gamma\overline{\beta}}=\delta_{\alpha\gamma}$, we have

$$\frac{\partial}{\partial t}g^{\alpha\overline{\beta}}=-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}\frac{\partial}{\partial t}g_{\gamma\overline{\delta}},$$

which together with (27) yields

$$\begin{aligned}\frac{\partial}{\partial t}g^{\alpha\overline{\beta}}=&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g_{\gamma\overline{\delta}}\\&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}g^{\sigma\overline{\eta}}g_{\xi\overline{\delta}}\widehat{R}_{\sigma\overline{\eta}\gamma\overline{\xi}}-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}g^{\sigma\overline{\eta}}g_{\gamma\overline{\xi}}\widehat{R}_{\sigma\overline{\eta}\xi\overline{\delta}}\\&+2g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}g^{\sigma\overline{\xi}}g^{\zeta\overline{\eta}}\widehat{\nabla}_{\gamma}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\delta}}g_{\sigma\overline{\eta}}.\end{aligned}\tag{28}$$

On the other hand, by (25) we get

$$\begin{aligned}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\alpha\overline{\beta}}=&-\widehat{\nabla}_{\overline{\zeta}}(g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}})\\=&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}-g^{\gamma\overline{\beta}}\widehat{\nabla}_{\overline{\zeta}}g^{\alpha\overline{\delta}}\cdot\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}\\&-g^{\alpha\overline{\delta}}\widehat{\nabla}_{\overline{\zeta}}g^{\gamma\overline{\beta}}\cdot\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}\\=&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}+g^{\gamma\overline{\beta}}g^{\alpha\overline{\theta}}g^{w\overline{\delta}}\widehat{\nabla}_{\overline{\zeta}}g_{w\overline{\theta}}\cdot\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}\\&+g^{\alpha\overline{\delta}}g^{\gamma\overline{\theta}}g^{w\overline{\beta}}\widehat{\nabla}_{\overline{\zeta}}g_{w\overline{\theta}}\cdot\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}},\\&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}=g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\alpha\overline{\beta}}\\&-g^{\gamma\overline{\beta}}g^{\alpha\overline{\theta}}g^{w\overline{\delta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}g_{w\overline{\theta}}\cdot\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}\\&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\theta}}g^{w\overline{\beta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}g_{w\overline{\theta}}\cdot\widehat{\nabla}_{\xi}g_{\gamma\overline{\delta}}.\end{aligned}\tag{29}$$

Similarly,

$$\begin{aligned}&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\beta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g_{\gamma\overline{\delta}}\\=&g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\alpha\overline{\beta}}-g^{\gamma\overline{\beta}}g^{\alpha\overline{\theta}}g^{w\overline{\delta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}g_{w\overline{\theta}}\cdot\widehat{\nabla}_{\overline{\zeta}}g_{\gamma\overline{\delta}}\\&-g^{\alpha\overline{\delta}}g^{\gamma\overline{\theta}}g^{w\overline{\beta}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}g_{w\overline{\theta}}\cdot\widehat{\nabla}_{\overline{\zeta}}g_{\gamma\overline{\delta}}.\end{aligned}\tag{30}$$

Substituting (29) and (30) into (28), we have

$$
\begin{aligned}
\frac{\partial}{\partial t} g^{\alpha\overline{\beta}} =& g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi} g^{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}} g^{\alpha\overline{\beta}} \\
& - g^{\alpha\overline{\delta}} g^{\gamma\overline{\theta}} g^{w\overline{\beta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\xi} g_{\gamma\overline{\delta}} \\
& - g^{\gamma\overline{\beta}} g^{\alpha\overline{\theta}} g^{w\overline{\delta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\xi} g_{\gamma\overline{\delta}} \\
& - g^{\alpha\overline{\delta}} g^{\gamma\overline{\theta}} g^{w\overline{\beta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\overline{\zeta}} g_{\gamma\overline{\delta}} \\
& - g^{\gamma\overline{\beta}} g^{\alpha\overline{\theta}} g^{w\overline{\delta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\overline{\zeta}} g_{\gamma\overline{\delta}} \\
& + 2 g^{\alpha\overline{\delta}} g^{\gamma\overline{\beta}} g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{\nabla}_{\gamma} g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\delta}} g_{\sigma\overline{\eta}} \\
& - g^{\alpha\overline{\delta}} g^{\sigma\overline{\eta}} \widehat{R}_{\sigma\overline{\eta}\beta\overline{\delta}} - g^{\gamma\overline{\beta}} g^{\sigma\overline{\eta}} \widehat{R}_{\sigma\overline{\eta}\gamma\overline{\alpha}}.
\end{aligned}
\tag{31}
$$

Combining (19) and (31) implies

$$
\begin{aligned}
\frac{\partial}{\partial t} g^{\alpha\overline{\beta}} =& g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi} g^{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}} g^{\alpha\overline{\beta}} \\
& - 2 g^{\alpha\overline{\delta}} g^{w\overline{\beta}} g^{\gamma\overline{\theta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} g_{\gamma\overline{\delta}} \cdot \widehat{\nabla}_{\overline{\zeta}} g_{w\overline{\theta}} \\
& - g^{\alpha\overline{\delta}} g^{\sigma\overline{\eta}} \widehat{R}_{\sigma\overline{\eta}\beta\overline{\delta}} - g^{\gamma\overline{\beta}} g^{\sigma\overline{\eta}} \widehat{R}_{\sigma\overline{\eta}\gamma\overline{\alpha}}.
\end{aligned}
\tag{32}
$$

For any two tensors A and B, let $A*B$ denote the linear combination of the tensor product of A and B. Let $\widehat{g}, \widehat{g}^{-1}, \widehat{Rm}, g, g^{-1}$ and Rm denote $\widehat{g}_{\alpha\overline{\beta}}, \widehat{g}^{\alpha\overline{\beta}}, \widehat{R}_{ijkl}, g_{\alpha\overline{\beta}}, g^{\alpha\overline{\beta}}$ and R_{ijkl} respectively. Let

$$
\begin{aligned}
g^2 =& g * g,\ g^3 = g * g * g, \dots, \\
g^{-2} =& g^{-1} * g^{-1},\ g^{-3} = g^{-1} * g^{-1} * g^{-1}, \dots .
\end{aligned}
\tag{33}
$$

Since $R_{ij} = g^{kl} R_{ikjl}$, the Ricci curvature can be denoted as $g^{-1} * Rm$. Thus (27) can be written as

$$
\begin{aligned}
\frac{\partial}{\partial t} g_{\alpha\overline{\beta}} =& g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi} g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}} g_{\alpha\overline{\beta}} \\
& - 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{\nabla}_{\alpha} g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} + g^{-1} * g * \widehat{g}^{-1} * \widehat{Rm}.
\end{aligned}
\tag{34}
$$

Differentiating both sides of (34) yields

$$
\begin{aligned}
\frac{\partial}{\partial t}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} =&\widehat{\nabla}_\gamma\left(\frac{\partial}{\partial t}g_{\alpha\overline{\beta}}\right) = \widehat{\nabla}_\gamma[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\xi g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\zeta}} g_{\alpha\overline{\beta}}\\
&- 2g^{\sigma\overline{\xi}}g^{\zeta\overline{\eta}}\widehat{\nabla}_\alpha g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\sigma\overline{\eta}} + g^{-1}*g*\widehat{g}^{-1}*\widehat{Rm}]\\
=&g^{\xi\overline{\zeta}}\widehat{\nabla}_\gamma\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\xi g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_\gamma\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\zeta}}g_{\alpha\overline{\beta}} + \widehat{\nabla}_\gamma g^{\xi\overline{\zeta}}\cdot\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\xi g_{\alpha\overline{\beta}}\\
&+ \widehat{\nabla}_\gamma g^{\xi\overline{\zeta}}\cdot\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\zeta}}g_{\alpha\overline{\beta}} - 2g^{\sigma\overline{\xi}}g^{\zeta\overline{\eta}}\widehat{\nabla}_\alpha g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_\gamma\widehat{\nabla}_{\overline{\beta}}g_{\sigma\overline{\eta}}\\
&- 2g^{\sigma\overline{\xi}}g^{\zeta\overline{\eta}}\widehat{\nabla}_{\overline{\beta}}g_{\sigma\overline{\eta}}\cdot\widehat{\nabla}_\gamma\widehat{\nabla}_\alpha g_{\zeta\overline{\xi}}\\
&- 2g^{\sigma\overline{\xi}}\cdot\widehat{\nabla}_\gamma g^{\zeta\overline{\eta}}\cdot\widehat{\nabla}_\alpha g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\sigma\overline{\eta}}\\
&- 2g^{\zeta\overline{\eta}}\cdot\widehat{\nabla}_\gamma g^{\sigma\overline{\xi}}\cdot\widehat{\nabla}_\alpha g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\sigma\overline{\eta}}\\
&+ \widehat{\nabla}_\gamma[g^{-1}*g*\widehat{g}^{-1}*\widehat{Rm}].
\end{aligned}
\tag{35}
$$

Using formulas (20) we obtain

$$
\begin{aligned}
\widehat{\nabla}_\gamma\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\xi g_{\alpha\overline{\beta}} =&\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\gamma\widehat{\nabla}_\xi g_{\alpha\overline{\beta}} + \widehat{R}_{\gamma\overline{\zeta}\xi\overline{\theta}}\widehat{\nabla}_\theta g_{\alpha\overline{\beta}}\\
&+ \widehat{R}_{\gamma\overline{\zeta}\alpha\overline{\theta}}\widehat{\nabla}_\xi g_{\theta\overline{\beta}} - \widehat{R}_{\gamma\overline{\zeta}\theta\overline{\beta}}\widehat{\nabla}_\xi g_{\alpha\overline{\theta}}\\
=&\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\gamma\widehat{\nabla}_\xi g_{\alpha\overline{\beta}} + \widehat{Rm}*\widehat{g}^{-1}*\widehat{\nabla}g\\
=&\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\xi\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + \widehat{Rm}*\widehat{g}^{-1}*\widehat{\nabla}g,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\widehat{\nabla}_\gamma\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\zeta}}g_{\alpha\overline{\beta}} =&\widehat{\nabla}_\xi\widehat{\nabla}_\gamma\widehat{\nabla}_{\overline{\zeta}}g_{\alpha\overline{\beta}}\\
=&\widehat{\nabla}_\xi[\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + \widehat{R}_{\gamma\overline{\zeta}\alpha\overline{\theta}}g_{\theta\overline{\beta}} - \widehat{R}_{\gamma\overline{\zeta}\theta\overline{\beta}}g_{\alpha\overline{\theta}}]\\
=&\widehat{\nabla}_\xi[\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + g*\widehat{g}^{-1}*\widehat{Rm}]\\
=&\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + \widehat{Rm}*\widehat{g}^{-1}*\widehat{\nabla}g\\
&+ g*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm},
\end{aligned}
\tag{37}
$$

which together with (35) yields

$$
\begin{aligned}
&g^{\xi\overline{\zeta}}\widehat{\nabla}_\gamma\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\xi g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_\gamma\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\zeta}}g_{\alpha\overline{\beta}}\\
&=g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\xi\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}\\
&\quad + g^{-1}*\widehat{Rm}*\widehat{g}^{-1}*\widehat{\nabla}g + g^{-1}*g*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm}.
\end{aligned}
\tag{38}
$$

From (25) it follows that

$$
\begin{aligned}
&\widehat{\nabla}_\gamma g^{\xi\overline{\zeta}} \cdot \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\xi g_{\alpha\overline{\beta}} + \widehat{\nabla}_\gamma g^{\xi\overline{\zeta}} \cdot \widehat{\nabla}_\xi \widehat{\nabla}_{\overline{\zeta}} g_{\alpha\overline{\beta}} \\
&\quad = -g^{\xi\overline{\theta}} g^{w\overline{\zeta}} \widehat{\nabla}_\gamma g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\xi g_{\alpha\overline{\beta}} - g^{\xi\overline{\theta}} g^{w\overline{\zeta}} \widehat{\nabla}_\gamma g_{w\overline{\theta}} \cdot \widehat{\nabla}_\xi \widehat{\nabla}_{\overline{\zeta}} g_{\alpha\overline{\beta}}.
\end{aligned} \tag{39}
$$

Substituting (38) and (39) into (35) implies

$$
\begin{aligned}
\frac{\partial}{\partial t} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} = & g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\xi \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_\xi \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} \\
& - g^{\xi\overline{\theta}} g^{w\overline{\zeta}} \widehat{\nabla}_\gamma g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\xi g_{\alpha\overline{\beta}} - g^{\xi\overline{\theta}} g^{w\overline{\zeta}} \widehat{\nabla}_\gamma g_{w\overline{\theta}} \cdot \widehat{\nabla}_\xi \widehat{\nabla}_{\overline{\zeta}} g_{\alpha\overline{\beta}} \\
& - 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{\nabla}_\alpha g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_\gamma \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} - 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} \cdot \widehat{\nabla}_\gamma \widehat{\nabla}_\alpha g_{\zeta\overline{\xi}} \\
& + 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\theta}} g^{w\overline{\eta}} \widehat{\nabla}_\gamma g_{w\overline{\theta}} \cdot \widehat{\nabla}_\alpha g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} \\
& + 2 g^{w\overline{\xi}} g^{\sigma\overline{\theta}} g^{\zeta\overline{\eta}} \widehat{\nabla}_\gamma g_{w\overline{\theta}} \cdot \widehat{\nabla}_\alpha g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} \\
& + g^{-1} * \widehat{Rm} * \widehat{g}^{-1} * \widehat{\nabla} g \\
& + g^{-1} * g * \widehat{g}^{-1} * \widehat{\nabla} \widehat{Rm} + \widehat{\nabla}_\gamma (g^{-1} * g * \widehat{g}^{-1} * \widehat{Rm}).
\end{aligned} \tag{40}
$$

We now define a function $\varphi(x,t)$ on $M \times [0,T]$:

$$
\varphi(x,t) = g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} \geq 0. \tag{41}
$$

Then

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t} = & 2 \operatorname{Re}\{ g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} \cdot \frac{\partial}{\partial t} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} \} \\
& + 2 \operatorname{Re} \left\{ \frac{\partial g^{\alpha\overline{\mu}}}{\partial t} \cdot g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} \right\} \\
& + g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} \cdot \frac{\partial g^{\gamma\overline{\lambda}}}{\partial t} \cdot \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}},
\end{aligned}
$$

which together with (32) and (40) implies

$$
\begin{aligned}
\frac{\partial\varphi}{\partial t} =& 2\operatorname{Re}\Big\{ g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} [g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \\
&\quad + g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \\
&\quad - g^{\xi\overline{\theta}} g^{w\overline{\zeta}} \widehat{\nabla}_{\gamma} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} g_{\alpha\overline{\beta}} - g^{\xi\overline{\theta}} g^{w\overline{\zeta}} \widehat{\nabla}_{\gamma} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} g_{\alpha\overline{\beta}} \\
&\quad - 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{\nabla}_{\alpha} g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\gamma} \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} - 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} \cdot \widehat{\nabla}_{\gamma} \widehat{\nabla}_{\alpha} g_{\zeta\overline{\xi}} \\
&\quad + 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\theta}} g^{w\overline{\eta}} \widehat{\nabla}_{\gamma} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\alpha} g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} \\
&\quad + 2 g^{\zeta\overline{\eta}} g^{\sigma\overline{\theta}} g^{w\overline{\xi}} \widehat{\nabla}_{\gamma} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\alpha} g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} \\
&\quad + g^{-1} * \widehat{Rm} * \widehat{g}^{-1} * \widehat{\nabla} g + g^{-1} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} \\
&\quad + \widehat{\nabla}_{\gamma} (g^{-1} * g * \widehat{g}^{-1} * \widehat{Rm})] \Big\} \\
&+ 2\operatorname{Re}\{ g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} [g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} g^{\alpha\overline{\mu}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} g^{\alpha\overline{\mu}} \\
&\quad - 2 g^{\alpha\overline{\delta}} g^{w\overline{\mu}} g^{\sigma\overline{\theta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\xi} g_{\sigma\overline{\delta}} + g^{-2} * \widehat{g}^{-1} * \widehat{Rm}] \} \\
&+ g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} \cdot [g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} g^{\gamma\overline{\lambda}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} g^{\gamma\overline{\lambda}} \\
&\quad - 2 g^{\gamma\overline{\delta}} g^{\sigma\overline{\theta}} g^{w\overline{\lambda}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} g_{w\overline{\theta}} \cdot \widehat{\nabla}_{\xi} g_{\sigma\overline{\delta}} + g^{-2} * \widehat{g}^{-1} * \widehat{Rm}].
\end{aligned} \tag{42}
$$

If we choose a coordinate system such that at one point

$$
\widehat{g}_{\alpha\overline{\beta}} = \delta_{\alpha\beta}, \quad (g_{\alpha\overline{\beta}}) = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}, \tag{43}
$$

then

$$
\widehat{g}^{\alpha\overline{\beta}} = \delta_{\alpha\beta}, \quad (g^{\alpha\overline{\beta}}) = \begin{pmatrix} \lambda_1^{-1} & & & 0 \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{-1} \end{pmatrix}, \tag{44}
$$

$$
\varphi(x,t) = \sum_{\alpha,\beta,\gamma} \frac{1}{\lambda_\alpha \lambda_\beta \lambda_\gamma} |\widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}}|^2. \tag{45}
$$

By (42) we get

$$
\begin{aligned}
\frac{\partial\varphi}{\partial t} =& 2\,\mathrm{Re}\{g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}]\} \\
&+ 2\,\mathrm{Re}\{g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\alpha\overline{\mu}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\alpha\overline{\mu}}]\} \\
&+ g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\gamma\overline{\lambda}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\gamma\overline{\lambda}}] \\
&+ 2\,\mathrm{Re}\Big\{-\frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\alpha\overline{\beta}} \\
&\qquad - \frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g_{\alpha\overline{\beta}} \\
&\qquad - \frac{2}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\alpha}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\gamma}\widehat{\nabla}_{\overline{\beta}}g_{\xi\overline{\zeta}} \\
&\qquad - \frac{2}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\gamma}\widehat{\nabla}_{\alpha}g_{\xi\overline{\zeta}} \\
&\qquad + 2\Phi_1 + 2\Phi_2 + g^{-3}*\widehat{\nabla}g*[g^{-1}*\widehat{Rm}*\widehat{g}^{-1}*\widehat{\nabla}g \\
&\qquad + g*g^{-1}*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm} + \widehat{\nabla}_{\gamma}(g^{-1}*g*\widehat{g}^{-1}*\widehat{Rm})]\Big\} \\
&+ 2\,\mathrm{Re}\{-2\Phi_3 + g^{-4}*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}g*\widehat{\nabla}g\} \\
&+ \{-2\Phi_4 + g^{-4}*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}g*\widehat{\nabla}g\},
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
\Phi_1 &= \frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta\lambda_\eta}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\eta\overline{\zeta}}\cdot\widehat{\nabla}_{\alpha}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\xi\overline{\eta}}, \\
\Phi_2 &= \frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta\lambda_\theta}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\xi\overline{\theta}}\cdot\widehat{\nabla}_{\alpha}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\theta\overline{\zeta}},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3 &= \frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\theta\lambda_\mu\lambda_\xi}\cdot\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\mu}}\cdot\widehat{\nabla}_{\overline{\xi}}g_{\mu\overline{\theta}}\cdot\widehat{\nabla}_{\xi}g_{\theta\overline{\alpha}}, \\
\Phi_4 &= \frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\theta\lambda_\lambda\lambda_\xi}\cdot\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\overline{\xi}}g_{\lambda\overline{\theta}}\cdot\widehat{\nabla}_{\xi}g_{\theta\overline{\gamma}}.
\end{aligned}
$$

Using the property (19) we obtain

$$
\Phi_1 = \Phi_2, \quad \Phi_3 = \Phi_4. \tag{47}
$$

From (22) it follows that

$$
\begin{aligned}
\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\alpha\overline{\beta}} =&\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g_{\alpha\overline{\beta}}+\widehat{Rm}*g\\
=&\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\beta}}g_{\alpha\overline{\zeta}}+\widehat{Rm}*g\\
=&\widehat{\nabla}_{\overline{\beta}}\widehat{\nabla}_{\xi}g_{\alpha\overline{\zeta}}+\widehat{Rm}*g\\
=&\widehat{\nabla}_{\overline{\beta}}\widehat{\nabla}_{\alpha}g_{\xi\overline{\zeta}}+\widehat{Rm}*g\\
=&\widehat{\nabla}_{\alpha}\widehat{\nabla}_{\overline{\beta}}g_{\xi\overline{\zeta}}+\widehat{Rm}*g.
\end{aligned}
\tag{48}
$$

Combining (46), (47) and (48) we get

$$
\begin{aligned}
\frac{\partial\varphi}{\partial t} =&2\,\mathrm{Re}\{g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}]\\
&\quad+g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\alpha\overline{\mu}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\alpha\overline{\mu}}]\}\\
&+g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\gamma\overline{\lambda}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\gamma\overline{\lambda}}]\\
&+\frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta}\mathrm{Re}\{-8\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\alpha\overline{\beta}}\\
&\qquad\qquad-4\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\xi\overline{\zeta}}\cdot\widehat{\nabla}_{\gamma}\widehat{\nabla}_{\alpha}g_{\zeta\overline{\xi}}\}\\
&+\frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta\lambda_\theta}\mathrm{Re}\{8\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\xi\overline{\theta}}\cdot\widehat{\nabla}_{\overline{\zeta}}g_{\theta\overline{\beta}}\cdot\widehat{\nabla}_{\zeta}g_{\alpha\overline{\xi}}\\
&\qquad\qquad-6\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\zeta}}\cdot\widehat{\nabla}_{\xi}g_{\theta\overline{\alpha}}\cdot\widehat{\nabla}_{\overline{\xi}}g_{\zeta\overline{\theta}}\}\\
&+g^{-4}*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}g*\widehat{\nabla}g+g^{-4}*g*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm}*\widehat{\nabla}g\\
&+g^{-3}*\widehat{\nabla}g*\widehat{\nabla}(g^{-1}*g*\widehat{g}^{-1}*\widehat{Rm}).
\end{aligned}
\tag{49}
$$

On the other hand, by the definition we have

$$
\begin{aligned}
\Delta\varphi =&g^{\xi\overline{\zeta}}\frac{\partial^2\varphi}{\partial z^\xi\partial\overline{z}^\zeta}+g^{\xi\overline{\zeta}}\frac{\partial^2\varphi}{\partial\overline{z}^\zeta\partial z^\xi}\\
=&g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\varphi+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\varphi\\
=&2\,\mathrm{Re}\{g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}]\\
&+g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\alpha\overline{\mu}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\alpha\overline{\mu}}]\}\\
&+g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\gamma\overline{\lambda}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\gamma\overline{\lambda}}]\\
&+2g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\\
&+2g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\\
&+8\,\mathrm{Re}[g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}g^{\alpha\overline{\mu}}\cdot\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}]
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&+ 8\,\mathrm{Re}[g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} g^{\alpha\overline{\mu}} \cdot \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}}]\\
&+ 4\,\mathrm{Re}[g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} g^{\gamma\overline{\lambda}} \cdot \widehat{\nabla}_{\xi} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}}]\\
&+ 4\,\mathrm{Re}[g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} g^{\gamma\overline{\lambda}} \cdot \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}}]\\
&+ 4\,\mathrm{Re}[g^{\gamma\overline{\lambda}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} g^{\alpha\overline{\mu}} \cdot \widehat{\nabla}_{\xi} g^{\nu\overline{\beta}} \cdot \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}}]\\
&+ 8\,\mathrm{Re}[g^{\nu\overline{\beta}} g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} g^{\alpha\overline{\mu}} \cdot \widehat{\nabla}_{\xi} g^{\gamma\overline{\lambda}} \cdot \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}}].
\end{aligned}
$$

Combining (25) and (44) gives

$$
\begin{cases}
\widehat{\nabla}_{\gamma} g^{\alpha\overline{\beta}} = -g^{\alpha\overline{\zeta}} g^{\xi\overline{\beta}} \widehat{\nabla}_{\gamma} g_{\xi\overline{\zeta}} = -\frac{1}{\lambda_\alpha\lambda_\beta} \widehat{\nabla}_{\gamma} g_{\beta\overline{\alpha}},\\
\widehat{\nabla}_{\overline{\gamma}} g^{\alpha\overline{\beta}} = -g^{\alpha\overline{\zeta}} g^{\xi\overline{\beta}} \widehat{\nabla}_{\overline{\gamma}} g_{\xi\overline{\zeta}} = -\frac{1}{\lambda_\alpha\lambda_\beta} \widehat{\nabla}_{\overline{\gamma}} g_{\beta\overline{\alpha}},
\end{cases}
\tag{51}
$$

which together with (50) implies

$$
\begin{aligned}
\Delta\varphi =& 2\,\mathrm{Re}\{g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} \cdot [g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}}]\\
&\qquad + g^{\nu\overline{\beta}} g^{\gamma\overline{\lambda}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} \cdot [g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} g^{\alpha\overline{\mu}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} g^{\alpha\overline{\mu}}]\}\\
&+ g^{\alpha\overline{\mu}} g^{\nu\overline{\beta}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\nu\overline{\mu}} \cdot [g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} g^{\gamma\overline{\lambda}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} g^{\gamma\overline{\lambda}}]\\
&+ \frac{2}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi} \widehat{\nabla}_{\overline{\xi}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\gamma}} g_{\beta\overline{\alpha}}\\
&+ \frac{2}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\xi}} \widehat{\nabla}_{\overline{\gamma}} g_{\beta\overline{\alpha}}\\
&+ \mathrm{Re}\Big\{-\frac{8}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\mu} \cdot \widehat{\nabla}_{\overline{\xi}} g_{\mu\overline{\alpha}} \cdot \widehat{\nabla}_{\overline{\gamma}} g_{\beta\overline{\mu}} \cdot \widehat{\nabla}_{\xi} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}}\\
&\qquad - \frac{8}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\mu} \cdot \widehat{\nabla}_{\xi} g_{\mu\overline{\alpha}} \cdot \widehat{\nabla}_{\overline{\gamma}} g_{\beta\overline{\mu}} \cdot \widehat{\nabla}_{\overline{\xi}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}}\\
&\qquad - \frac{4}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\lambda} \cdot \widehat{\nabla}_{\overline{\xi}} g_{\lambda\overline{\gamma}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\beta\overline{\alpha}} \cdot \widehat{\nabla}_{\xi} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}}\\
&\qquad - \frac{4}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\lambda} \cdot \widehat{\nabla}_{\xi} g_{\lambda\overline{\gamma}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\beta\overline{\alpha}} \cdot \widehat{\nabla}_{\overline{\xi}} \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}}\\
&\qquad + \frac{4}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\mu\lambda_\nu} \cdot \widehat{\nabla}_{\overline{\xi}} g_{\mu\overline{\alpha}} \cdot \widehat{\nabla}_{\xi} g_{\beta\overline{\nu}} \cdot \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\gamma}} g_{\nu\overline{\mu}}\\
&\qquad + \frac{8}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\mu\lambda_\lambda} \cdot \widehat{\nabla}_{\overline{\xi}} g_{\mu\overline{\alpha}} \cdot \widehat{\nabla}_{\xi} g_{\lambda\overline{\gamma}} \cdot \widehat{\nabla}_{\gamma} g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\overline{\lambda}} g_{\beta\overline{\mu}}\Big\},
\end{aligned}
\tag{52}
$$

which together with (19) yields

$$
\begin{aligned}
\Delta\varphi =& 2\,\mathrm{Re}\{g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}]\\
&\qquad+g^{\nu\overline{\beta}}g^{\gamma\overline{\lambda}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\alpha\overline{\mu}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\alpha\overline{\mu}}]\}\\
&+g^{\alpha\overline{\mu}}g^{\nu\overline{\beta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\lambda}}g_{\nu\overline{\mu}}\cdot[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g^{\gamma\overline{\lambda}}+g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}g^{\gamma\overline{\lambda}}]\\
&+\frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi}[2\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}+2\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}]\\
&+\frac{1}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\mu}\,\mathrm{Re}\{-12\widehat{\nabla}_{\overline{\xi}}g_{\mu\overline{\gamma}}\cdot\widehat{\nabla}_{\overline{\mu}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\\
&\qquad\qquad-12\widehat{\nabla}_{\xi}g_{\mu\overline{\gamma}}\cdot\widehat{\nabla}_{\overline{\mu}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\}\\
&+\frac{12}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\mu\lambda_\nu}\,\mathrm{Re}[\widehat{\nabla}_{\overline{\xi}}g_{\mu\overline{\alpha}}\cdot\widehat{\nabla}_{\xi}g_{\beta\overline{\nu}}\cdot\widehat{\nabla}_{\overline{\gamma}}g_{\nu\overline{\mu}}\cdot\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}].
\end{aligned}
\tag{53}
$$

Combining (49) and (53), we get

$$
\begin{aligned}
\frac{\partial\varphi}{\partial t} =& \Delta\varphi-\frac{2}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi}[\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\\
&\qquad+\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}]\\
&+\frac{2}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta}\,\mathrm{Re}\{2\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\zeta\overline{\xi}}\cdot\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}g_{\alpha\overline{\beta}}\\
&\qquad+4\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\overline{\beta}}g_{\xi\overline{\zeta}}\cdot\widehat{\nabla}_{\gamma}\widehat{\nabla}_{\alpha}g_{\zeta\overline{\xi}}\}\\
&+\frac{2}{\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\xi\lambda_\zeta\lambda_\theta}\,\mathrm{Re}\{-2\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\alpha}}\cdot\widehat{\nabla}_{\gamma}g_{\xi\overline{\theta}}\cdot\widehat{\nabla}_{\overline{\zeta}}g_{\theta\overline{\beta}}\cdot\widehat{\nabla}_{\zeta}g_{\alpha\overline{\xi}}\\
&\qquad-3\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\cdot\widehat{\nabla}_{\overline{\gamma}}g_{\beta\overline{\zeta}}\cdot\widehat{\nabla}_{\xi}g_{\theta\overline{\alpha}}\cdot\widehat{\nabla}_{\overline{\xi}}g_{\zeta\overline{\theta}}\}\\
&+g^{-4}*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}g*\widehat{\nabla}g\\
&+g^{-4}*g*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm}*\widehat{\nabla}g\\
&+g^{-3}*\widehat{\nabla}g*\widehat{\nabla}[g^{-1}*g*\widehat{g}^{-1}*\widehat{Rm}].
\end{aligned}
\tag{54}
$$

Lemma 7.3. *We have*

$$
\begin{aligned}
\frac{\partial\varphi}{\partial t} =&\Delta\varphi - \frac{2}{\lambda_\alpha\lambda_\beta\lambda_\xi\lambda_\gamma}|\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} - \frac{1}{\lambda_\zeta}\widehat{\nabla}_{\overline{\zeta}} g_{\gamma\overline{\xi}} \cdot \widehat{\nabla}_\zeta g_{\alpha\overline{\beta}}|^2 \\
&- \frac{2}{\lambda_\alpha\lambda_\beta\lambda_\xi\lambda_\gamma}|\widehat{\nabla}_\xi\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} - \frac{1}{\lambda_\zeta}\widehat{\nabla}_\xi g_{\gamma\overline{\zeta}} \cdot \widehat{\nabla}_\alpha g_{\zeta\overline{\beta}} - \frac{1}{\lambda_\zeta}\widehat{\nabla}_\alpha g_{\gamma\overline{\zeta}} \cdot \widehat{\nabla}_\xi g_{\zeta\overline{\beta}}|^2 \\
&+ g^{-4} * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla} g * \widehat{\nabla} g + g^{-4} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} * \widehat{\nabla} g \\
&+ g^{-3} * \widehat{\nabla} g * \widehat{\nabla}[g^{-1} * g * \widehat{g}^{-1} * \widehat{Rm}].
\end{aligned} \tag{55}
$$

We now define a function $q(t)$:

$$
q(t) = e^{C_2(t+2)^{\frac{2-\theta}{\theta}}}, \quad 0 \le t \le T, \tag{56}
$$

where C_2 is the constant in Lemma 7.2. From (13) in Lemma 7.2 it follows that

$$
ds_0^2 \ge ds_t^2 \ge \frac{1}{q(t)}\, ds_0^2, \quad 0 \le t \le T. \tag{57}
$$

Since $\dfrac{\partial}{\partial t} g_{ij} = -2R_{ij} \le 0$, we have

$$
ds_0^2 \ge ds_{\tau_0}^2 \ge ds_t^2, \quad \tau_0 \le t \le T, \tag{58}
$$

which together with (57) implies

$$
ds_{\tau_0}^2 \ge ds_t^2 \ge \frac{1}{q(t)}\, ds_{\tau_0}^2, \quad \tau_0 \le t \le T. \tag{59}
$$

Thus

$$
\widehat{g}_{\alpha\overline{\beta}} \ge g_{\alpha\overline{\beta}} \ge \frac{1}{q(t)}\widehat{g}_{\alpha\overline{\beta}}, \quad \tau_0 \le t \le T, \tag{60}
$$

$$
\frac{1}{q(t)} \le \lambda_\alpha \le 1, \tag{61}
$$

$$
\widehat{g}^{\alpha\overline{\beta}} \le g^{\alpha\overline{\beta}} \le q(t)\widehat{g}^{\alpha\overline{\beta}}, \quad \tau_0 \le t \le T. \tag{62}
$$

From (15), (60), (61) and (62) we get the estimates of the terms in (55):

$$\begin{aligned} & g^{-4} * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla} g * \widehat{\nabla} g \leq C_3(n,k_0) \cdot q(t)^4 |\widehat{\nabla} g|^2 \\ (63) \qquad & \leq\ C_3(n,k_0) q(t)^4 \varphi(x,t), \quad \tau_0 \leq t \leq T, \\ & g^{-4} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} * \widehat{\nabla} g \leq C_4(n,k_0) q(t)^4 |\widehat{\nabla} g| \\ (64) \qquad & \leq\ C_4(n,k_0) q(t)^4 \varphi(x,t)^{\frac{1}{2}}, \quad \tau_0 \leq t \leq T, \end{aligned}$$

where $0 < C_3(n,k_0),\ C_4(n,k_0) < +\infty$ are the constants depending only on n and k_0. Since $\widehat{\nabla} g^{-1} = g^{-2} * \widehat{\nabla} g$, we have

$$\begin{aligned} & g^{-3} * \widehat{\nabla} g * \widehat{\nabla}[g^{-1} * g * \widehat{g}^{-1} * \widehat{Rm}] \\ & \qquad = g^{-5} * g * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla} g * \widehat{\nabla} g \\ & \qquad\quad + g^{-4} * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla} g * \widehat{\nabla} g \\ (65) \qquad & \qquad\quad + g^{-4} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} * \widehat{\nabla} g \\ & \qquad \leq C_5(n,k_0) q(t)^5 |\widehat{\nabla} g|^2 + C_5(n,k_0) q(t)^4 |\widehat{\nabla} g|^2 \\ & \qquad\quad + C_5(n,k_0) q(t)^4 |\widehat{\nabla} g| \\ & \qquad \leq C_6(n,k_0) q(t)^5 \varphi(x,t) + C_5(n,k_0) q(t)^4 \varphi(x,t)^{\frac{1}{2}}, \\ & \qquad\qquad\qquad\qquad \tau_0 \leq t \leq T, \end{aligned}$$

where $0 < C_5(n,k_0),\ C_6(n,k_0) < +\infty$ are constants depending only on n and k_0. Combining (55), (63), (64) and (65) yields

$$\begin{aligned} & \frac{\partial \varphi}{\partial t} \leq \Delta\varphi + C_7(n,k_0) q(t)^5 \varphi(x,t) \\ (66) \qquad & \qquad + C_7(n,k_0) q(t)^4 \varphi(x,t)^{\frac{1}{2}}, \quad \tau_0 \leq t \leq T, \\ & \frac{\partial \varphi}{\partial t} \leq \Delta\varphi + C_8(n,k_0) q(t)^5 \varphi(x,t) \\ (67) \qquad & \qquad + C_8(n,k_0) q(t)^3, \quad \tau_0 \leq t \leq T, \end{aligned}$$

where $0 < C_7(n,k_0),\ C_8(n,k_0) < +\infty$ are constants depending only on n and k_0. On the other hand, from (27) it follows that

$$\begin{aligned} & \frac{\partial}{\partial t}[\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] = g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_{\xi} [\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] \\ & \qquad + g^{\xi\overline{\zeta}} \widehat{\nabla}_{\xi} \widehat{\nabla}_{\overline{\zeta}} [\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] + g^{\sigma\overline{\eta}} g_{\xi\overline{\beta}} \widehat{g}^{\alpha\overline{\beta}} \widehat{R}_{\sigma\overline{\eta}\alpha\overline{\xi}} \\ (68) \qquad & \qquad + g^{\sigma\overline{\eta}} g_{\alpha\overline{\xi}} \widehat{g}^{\alpha\overline{\beta}} \widehat{R}_{\sigma\overline{\eta}\xi\overline{\beta}} - 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{g}^{\alpha\overline{\beta}} \widehat{\nabla}_{\alpha} g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}} \\ & \quad = \Delta[\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] + g^{\sigma\overline{\eta}} g_{\xi\overline{\beta}} \widehat{g}^{\alpha\overline{\beta}} \widehat{R}_{\sigma\overline{\eta}\alpha\overline{\xi}} \\ & \qquad + g^{\sigma\overline{\eta}} g_{\alpha\overline{\xi}} \widehat{g}^{\alpha\overline{\beta}} \widehat{R}_{\sigma\overline{\eta}\xi\overline{\beta}} - 2 g^{\sigma\overline{\xi}} g^{\zeta\overline{\eta}} \widehat{g}^{\alpha\overline{\beta}} \widehat{\nabla}_{\alpha} g_{\zeta\overline{\xi}} \cdot \widehat{\nabla}_{\overline{\beta}} g_{\sigma\overline{\eta}}. \end{aligned}$$

Combining (11), (62) and (68) we get

$$\frac{\partial}{\partial t}[\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] \leq \Delta[\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] - \frac{2}{q(t)}\varphi(x,t), \quad \text{on } M \times [\tau_0, T]. \tag{69}$$

On the other hand, combining (10) and Lemma 2.3 shows that there exists a constant $0 < C_9 < +\infty$ depending only on n such that

$$\sup_{x \in M} |\nabla R_{ijkl}(x,t)|^2 \leq C_9 \left[\frac{\Theta}{t} + \Theta^{\frac{3}{2}}\right], \quad 0 \leq t \leq T. \tag{70}$$

From (14) we know that τ_0 depends only on n and k_0, so that, in consequence of (70),

$$\sup_{x \in M} |\nabla R_{ijkl}(x,t)| \leq C_{10}, \quad \tau_0 \leq t \leq T, \tag{71}$$

where $0 < C_{10} < +\infty$ is a constant depending only on n, k_0 and Θ. By the definition we have

$$\begin{aligned} \frac{\partial}{\partial t}\widehat{\nabla}_k g_{ij} &= \widehat{\nabla}_k \left(\frac{\partial}{\partial t} g_{ij}\right) = -2\widehat{\nabla}_k R_{ij} \\ &= -2\nabla_k R_{ij} + Rm * g^{-1} * g^{-1} * \widehat{\nabla} g, \end{aligned} \tag{72}$$

which together with (10), (62) and (71) implies

$$\left|\frac{\partial}{\partial t}\widehat{\nabla} g\right| \leq C_{11} + C_{11} q(t)^2 |\widehat{\nabla} g|, \quad \tau_0 \leq t \leq T, \tag{73}$$

$$\left|\frac{\partial}{\partial t}\widehat{\nabla} g\right| \leq C_{11} + C_{11} q(T)^2 |\widehat{\nabla} g| \leq C_{12} + C_{12}|\widehat{\nabla} g|, \quad \tau_0 \leq t \leq T, \tag{74}$$

where $0 < C_{11} < +\infty$ is a constant depending only on n, k_0 and Θ, and $0 < C_{12} < +\infty$ is a constant depending only on n, k_0, θ, C_1, T and Θ. By the definition we know that

$$\widehat{\nabla}_k g_{ij}(x, \tau_0) \equiv 0, \quad \forall\ x \in M, \tag{75}$$

which together with (74) implies

$$\sup_{x \in M} |\widehat{\nabla} g(x,t)| \leq C_{13} e^{C_{12} t} \leq C_{13} e^{C_{12} T} \leq C_{14}, \ \tau_0 \leq t \leq T, \tag{76}$$

where $0 < C_{13}$, $C_{14} < +\infty$ are constants depending only on n, k_0, θ, C_1, T and Θ. Combining (41) and (62) we get

$$\begin{aligned}\varphi(x,t) &\leq q(t)^3|\widehat{\nabla} g(x,t)|^2 \leq C_{14}^2 q(t)^3 \\ &\leq C_{14}^2 q(T)^3 \leq C_{15}, \quad x \in M,\ \tau_0 \leq t \leq T,\end{aligned} \tag{77}$$

where $0 < C_{15} < +\infty$ is a constant depending only on n, k_0, θ, C_1, T and Θ. For any $t_0 \in [\tau_0, T]$, from (67) and (69) it follows that

$$\frac{\partial \varphi}{\partial t} \leq \Delta\varphi + C_8 q(t_0)^5 \varphi(x,t) + C_8 q(t_0)^3,\ x \in M,\ \tau_0 \leq t \leq t_0, \tag{78}$$

$$\frac{\partial}{\partial t}[\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] \leq \Delta[\widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}}] - \frac{2}{q(t_0)}\varphi(x,t),\ x \in M,\ \tau_0 \leq t \leq t_0. \tag{79}$$

Now we define a function $\Phi(x,t)$ on $M \times [\tau_0, t_0]$:

$$\Phi(x,t) = \varphi(x,t) + C_8 q(t_0)^6 \widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}} - C_8 q(t_0)^3 t. \tag{80}$$

By (78) and (79) we obtain

$$\frac{\partial}{\partial t}\Phi(x,t) \leq \Delta\Phi(x,t), \quad \text{on } M \times [\tau_0, t_0]. \tag{81}$$

Combining (60), (77) and (80) yields

$$\begin{aligned}\Phi(x,t) &\leq \varphi(x,t) + C_8 q(t_0)^6 \widehat{g}^{\alpha\overline{\beta}} g_{\alpha\overline{\beta}} \\ &\leq C_{15} + nC_8 q(t_0)^6, \quad \text{on } M \times [\tau_0, t_0].\end{aligned} \tag{82}$$

From (75) and (80) we know that

$$\begin{aligned}\Phi(x,\tau_0) &\leq \varphi(x,\tau_0) + C_8 q(t_0)^6 \widehat{g}^{\alpha\overline{\beta}} \widehat{g}_{\alpha\overline{\beta}} \\ &\leq 0 + nC_8 q(t_0)^6 = nC_8 q(t_0)^6, \quad x \in M.\end{aligned} \tag{83}$$

Using Theorem 4.8 from (81), (82) and (83) we get

$$\Phi(x,t) \leq nC_8 q(t_0)^6, \quad \text{on } M \times [\tau_0, t_0]. \tag{84}$$

Combining (80) and (84) leads to

$$\varphi(x,t) \leq nC_8 q(t_0)^6 + C_8 q(t_0)^3 t, \quad \text{on } M \times [\tau_0, t_0]. \tag{85}$$

$$\varphi(x,t_0) \leq nC_8 q(t_0)^6 + C_8 q(t_0)^3 t_0 \leq C_{16} q(t_0)^6, \quad x \in M, \tag{86}$$

where $0 < C_{16} < +\infty$ is a constant depending only on n, k_0, θ and C_1. Since $t_0 \in [\tau_0, T]$ is arbitrary, we have

$$\sup_{x \in M} \varphi(x,t) \leq C_{16} q(t)^6, \quad \tau_0 \leq t \leq T. \tag{87}$$

Combining (62) and (87) we get

Lemma 7.4. *There exists a constant* $0 < C_{16} < +\infty$ *depending only on* n, k_0, θ *and* C_1 *such that*

$$\sup_{x \in M} |\widehat{\nabla} g(x,t)|^2 \leq C_{16} q(t)^6, \quad \tau_0 \leq t \leq T. \tag{88}$$

Now we want to estimate the second order covariant derivatives $|\widehat{\nabla}\widehat{\nabla} g|^2$. From Lemma 7.3 we know that there exists a constant $0 < C_{17}(n) < +\infty$ depending only on n such that

$$\begin{aligned} \frac{\partial \varphi}{\partial t} \leq & \Delta\varphi - \frac{1}{\lambda_\alpha \lambda_\beta \lambda_\gamma \lambda_\xi}[|\widehat{\nabla}_{\overline{\xi}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 + |\widehat{\nabla}_\xi \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2] \\ & + C_{17}(n)\varphi(x,t)^2 + g^{-4} * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla} g * \widehat{\nabla} g \\ & + g^{-4} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} * \widehat{\nabla} g \\ & + g^{-3} * \widehat{\nabla} g * \widehat{\nabla}[g^{-1} * g * \widehat{g}^{-1} * \widehat{Rm}]. \end{aligned} \tag{89}$$

which together with (63), (64), (65) and (87) implies

$$\begin{aligned} \frac{\partial \varphi}{\partial t} \leq & \Delta\varphi - \frac{1}{\lambda_\alpha \lambda_\beta \lambda_\gamma \lambda_\xi}[|\widehat{\nabla}_{\overline{\xi}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 + |\widehat{\nabla}_\xi \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2] \\ & + C_{18} q(t)^{12}, \quad \tau_0 \leq t \leq T, \end{aligned} \tag{90}$$

where $0 < C_{18} < +\infty$ is a constant depending only on n, k_0, θ and C_1. On the other hand, (40) can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} = & g^{\xi\overline{\zeta}} \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\xi \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}} \widehat{\nabla}_\xi \widehat{\nabla}_{\overline{\zeta}} \widehat{\nabla}_\gamma g_{\alpha\overline{\beta}} \\ & + g^{-2} * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g + g^{-3} * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla} g \\ & + g^{-1} * \widehat{\nabla} g * \widehat{g}^{-1} * \widehat{Rm} + g^{-1} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} \\ & + \widehat{\nabla}_\gamma [g^{-1} * g * \widehat{g}^{-1} * \widehat{Rm}]. \end{aligned} \tag{91}$$

Differentiating both sides of (91) yields

$$
\begin{aligned}
\frac{\partial}{\partial t}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} =& \widehat{\nabla}_{\overline{\delta}}\left(\frac{\partial}{\partial t}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\right)\\
=& \widehat{\nabla}_{\overline{\delta}}[g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\\
&+ g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + g^{-2}*\widehat{\nabla}g*\widehat{\nabla}\widehat{\nabla}g\\
&+ g^{-3}*\widehat{\nabla}g*\widehat{\nabla}g*\widehat{\nabla}g + g^{-1}*\widehat{\nabla}g*\widehat{g}^{-1}*\widehat{Rm}\\
&+ g^{-1}*g*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm} + \widehat{\nabla}_{\gamma}(g^{-1}*g*\widehat{g}^{-1}*\widehat{Rm})].
\end{aligned}
\tag{92}
$$

From (20) it follows that

$$
\begin{aligned}
\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} =& \widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\\
=& \widehat{\nabla}_{\overline{\zeta}}[\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + \widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}g]\\
=& \widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + \widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm}*\widehat{\nabla}g\\
&+ \widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}\widehat{\nabla}g,
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} =& \widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + \widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}\widehat{\nabla}g\\
=& \widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + \widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}\widehat{\nabla}g.
\end{aligned}
\tag{94}
$$

Combining (25), (92), (93) and (94), we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} =& g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\\
&+ g^{-2}*\widehat{\nabla}g*\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g\\
&+ g^{-1}*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}\widehat{\nabla}g\\
&+ g^{-1}*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm}*\widehat{\nabla}g + g^{-2}*\widehat{\nabla}\widehat{\nabla}g*\widehat{\nabla}\widehat{\nabla}g\\
&+ g^{-3}*\widehat{\nabla}g*\widehat{\nabla}g*\widehat{\nabla}\widehat{\nabla}g\\
&+ g^{-4}*\widehat{\nabla}g*\widehat{\nabla}g*\widehat{\nabla}g*\widehat{\nabla}g\\
&+ g^{-2}*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}g*\widehat{\nabla}g\\
&+ g^{-2}*g*\widehat{g}^{-1}*\widehat{\nabla}\widehat{Rm}*\widehat{\nabla}g\\
&+ g^{-2}*g*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}\widehat{\nabla}g\\
&+ g^{-1}*g*\widehat{g}^{-1}*\widehat{\nabla}\widehat{\nabla}\widehat{Rm}\\
&+ g^{-3}*g*\widehat{g}^{-1}*\widehat{Rm}*\widehat{\nabla}g*\widehat{\nabla}g,
\end{aligned}
\tag{95}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} =& g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}\\
&+ g^{-2} * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g + g^{-2} * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g\\
&+ g^{-3} * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g\\
&+ g^{-4} * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla} g\\
&+ g^{-1} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{\nabla}\widehat{Rm} + g^{-2} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} * \widehat{\nabla} g\\
&+ g^{-2} * g * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla}\widehat{\nabla} g\\
&+ g^{-3} * g * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla} g * \widehat{\nabla} g.
\end{aligned}
\tag{96}
$$

On the other hand, (32) can be written as

$$
\begin{aligned}
\frac{\partial}{\partial t} g^{\alpha\overline{\beta}} =& g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi} g^{\alpha\overline{\beta}} + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}} g^{\alpha\overline{\beta}}\\
&+ g^{-4} * \widehat{\nabla} g * \widehat{\nabla} g + g^{-2} * \widehat{g}^{-1} * \widehat{Rm}.
\end{aligned}
\tag{97}
$$

In the remainder of this section, we always use $|\ |^2$ to denote the norm with respect to ds_t^2. We have

$$
\begin{aligned}
|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 &= g^{\alpha\overline{\sigma}}g^{\eta\overline{\beta}}g^{\gamma\overline{\theta}}g^{\lambda\overline{\delta}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}} \cdot \widehat{\nabla}_{\lambda}\widehat{\nabla}_{\overline{\theta}}g_{\eta\overline{\sigma}}\\
&= g^{-4} * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g,
\end{aligned}
\tag{98}
$$

which together with (96) and (97) implies

$$
\begin{aligned}
\frac{\partial}{\partial t}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 =& g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2\\
&- 2|\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 - 2|\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2\\
&+ g^{-6} * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g\\
&+ g^{-7} * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g\\
&+ g^{-3} * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g * [g^{-4} * \widehat{\nabla} g * \widehat{\nabla} g + g^{-2} * \widehat{g}^{-1} * \widehat{Rm}]\\
&+ g^{-4} * \widehat{\nabla}\widehat{\nabla} g * [g^{-2} * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g + g^{-2} * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g\\
&+ g^{-3} * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g + g^{-4} * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla} g\\
&+ g^{-1} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{\nabla}\widehat{Rm} + g^{-2} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} * \widehat{\nabla} g\\
&+ g^{-2} * g * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla}\widehat{\nabla} g\\
&+ g^{-3} * g * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla} g * \widehat{\nabla} g],
\end{aligned}
\tag{99}
$$

where we have used the similar arguments as what we did in the proof of (54). By the definition we obtain

$$\begin{aligned}\Delta|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 &= g^{\xi\overline{\zeta}}\frac{\partial^2}{\partial\overline{z}^{\zeta}\partial z^{\xi}}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 + g^{\xi\overline{\zeta}}\frac{\partial^2}{\partial z^{\xi}\partial\overline{z}^{\zeta}}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 \\ &= g^{\xi\overline{\zeta}}\widehat{\nabla}_{\overline{\zeta}}\widehat{\nabla}_{\xi}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 + g^{\xi\overline{\zeta}}\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\zeta}}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2. \qquad (100)\end{aligned}$$

Using (15), (60), (61), (62) and Lemma 7.4 we get

$$g^{-6} * \widehat{\nabla}g * \widehat{\nabla}\widehat{\nabla}g * \widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g \le C_{19}q(t)^9|\widehat{\nabla}\widehat{\nabla}g|\cdot|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g|, \quad \tau_0 \le t \le T, \qquad (101)$$

$$g^{-6} * \widehat{\nabla}\widehat{\nabla}g * \widehat{\nabla}\widehat{\nabla}g * \widehat{\nabla}\widehat{\nabla}g \le C_{20}q(t)^6|\widehat{\nabla}\widehat{\nabla}g|^3, \quad \tau_0 \le t \le T, \qquad (102)$$

$$g^{-7} * \widehat{\nabla}g * \widehat{\nabla}g * \widehat{\nabla}\widehat{\nabla}g * \widehat{\nabla}\widehat{\nabla}g \le C_{21}q(t)^{13}|\widehat{\nabla}\widehat{\nabla}g|^2, \quad \tau_0 \le t \le T, \qquad (103)$$

$$g^{-8} * \widehat{\nabla}g * \widehat{\nabla}g * \widehat{\nabla}g * \widehat{\nabla}g * \widehat{\nabla}\widehat{\nabla}g \le C_{22}q(t)^{20}|\widehat{\nabla}\widehat{\nabla}g|, \quad \tau_0 \le t \le T, \qquad (104)$$

$$\begin{aligned}&g^{-4} * \widehat{\nabla}\widehat{\nabla}g * [g^{-1} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{\nabla}\widehat{Rm} \\ &\qquad + g^{-2} * g * \widehat{g}^{-1} * \widehat{\nabla}\widehat{Rm} * \widehat{\nabla}g \\ &\quad + g^{-2} * g * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla}\widehat{\nabla}g \\ &\qquad + g^{-3} * g * \widehat{g}^{-1} * \widehat{Rm} * \widehat{\nabla}g * \widehat{\nabla}g] \\ &\le C_{23}q(t)^{13}|\widehat{\nabla}\widehat{\nabla}g| + C_{24}q(t)^6|\widehat{\nabla}\widehat{\nabla}g|^2, \quad \tau_0 \le t \le T, \qquad (105)\end{aligned}$$

where $0 < C_{19}, C_{20}, C_{21}, C_{22}, C_{23}, C_{24} < +\infty$ are constants depending only on n, k_0, θ and C_1. Now substituting (100)-(105) into (99), we have

$$\begin{aligned}\frac{\partial}{\partial t}|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 \le &\Delta|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 - 2|\widehat{\nabla}_{\xi}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 \\ &- 2|\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\gamma}g_{\alpha\overline{\beta}}|^2 + C_{25}q(t)^9|\widehat{\nabla}\widehat{\nabla}g|\cdot|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g| \\ &+ C_{26}q(t)^6|\widehat{\nabla}\widehat{\nabla}g|^3 + C_{27}q(t)^{13}|\widehat{\nabla}\widehat{\nabla}g|^2 \\ &+ C_{28}q(t)^{20}|\widehat{\nabla}\widehat{\nabla}g|, \quad \tau_0 \le t \le T, \qquad (106)\end{aligned}$$

where $0 < C_{25}, C_{26}, C_{27}, C_{28} < +\infty$ are constants depending only on n, k_0, θ and C_1. Similarly,

$$
\begin{aligned}
\frac{\partial}{\partial t}|\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 \leq & \Delta|\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 - 2|\widehat{\nabla}_\xi\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 \\
& - 2|\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 \\
& + C_{25}q(t)^9|\widehat{\nabla}\widehat{\nabla}g|\cdot|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g| \\
& + C_{26}q(t)^6|\widehat{\nabla}\widehat{\nabla}g|^3 + C_{27}q(t)^{13}|\widehat{\nabla}\widehat{\nabla}g|^2 \\
& + C_{28}q(t)^{20}|\widehat{\nabla}\widehat{\nabla}g|, \quad \tau_0 \leq t \leq T.
\end{aligned} \tag{107}
$$

By the definition we know that (where $A,\ B,\ C,\ D = \alpha$ or $\overline{\alpha}$)

$$
\begin{aligned}
|\widehat{\nabla}\widehat{\nabla}g|^2 = & |\widehat{\nabla}_C\widehat{\nabla}_D g_{AB}|^2 = 2|\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 + 2|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 \\
& + 2|\widehat{\nabla}_\delta\widehat{\nabla}_{\overline{\gamma}} g_{\alpha\overline{\beta}}|^2 + 2|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_{\overline{\gamma}} g_{\alpha\overline{\beta}}|^2 \\
= & 4|\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 + 4|\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2,
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g|^2 = & 4[|\widehat{\nabla}_\xi\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 + |\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_{\overline{\delta}}\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 \\
& + |\widehat{\nabla}_{\overline{\xi}}\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 + |\widehat{\nabla}_\xi\widehat{\nabla}_\delta\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2].
\end{aligned} \tag{109}
$$

Combining (106), (107), (108) and (109) we get

$$
\begin{aligned}
\frac{\partial}{\partial t}|\widehat{\nabla}\widehat{\nabla}g|^2 \leq & \Delta|\widehat{\nabla}\widehat{\nabla}g|^2 - 2|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g|^2 \\
& + 8C_{25}q(t)^9|\widehat{\nabla}\widehat{\nabla}g|\cdot|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g| + 8C_{26}q(t)^6|\widehat{\nabla}\widehat{\nabla}g|^3 \\
& + 8C_{27}q(t)^{13}|\widehat{\nabla}\widehat{\nabla}g|^2 + 8C_{28}q(t)^{20}|\widehat{\nabla}\widehat{\nabla}g|, \\
& \qquad \tau_0 \leq t \leq T,
\end{aligned} \tag{110}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial t}|\widehat{\nabla}\widehat{\nabla}g|^2 \\
& \leq \Delta|\widehat{\nabla}\widehat{\nabla}g|^2 - |\widehat{\nabla}\widehat{\nabla}\widehat{\nabla}g|^2 + C_{29}q(t)^{12}|\widehat{\nabla}\widehat{\nabla}g|^3 + C_{30}q(t)^{30}, \\
& \qquad \tau_0 \leq t \leq T,
\end{aligned} \tag{111}
$$

where $0 < C_{29}, C_{30} < +\infty$ are constants depending only on n, k_0, θ and C_1.

Similar to (108) we have

$$
|\widehat{\nabla}g|^2 = 4|\widehat{\nabla}_\gamma g_{\alpha\overline{\beta}}|^2 = 4\varphi(x,t); \tag{112}
$$

thus from (90) it follows that

$$
\frac{\partial}{\partial t}|\widehat{\nabla}g|^2 \leq \Delta|\widehat{\nabla}g|^2 - |\widehat{\nabla}\widehat{\nabla}g|^2 + 4C_{18}q(t)^{12}, \quad \tau_0 \leq t \leq T. \tag{113}
$$

Suppose $a > 0$ is a constant to be determined later. Then

$$\frac{\partial}{\partial t}[a + |\widehat{\nabla} g|^2] \le \Delta[a + |\widehat{\nabla} g|^2] - |\widehat{\nabla}\widehat{\nabla} g|^2 + 4C_{18} q(t)^{12}, \quad \tau_0 \le t \le T. \tag{114}$$

Now we define a new function

$$\psi(x,t) = [a + |\widehat{\nabla} g|^2] \cdot |\widehat{\nabla}\widehat{\nabla} g|^2. \tag{115}$$

Then from (111) and (114) we obtain

$$\begin{aligned} \frac{\partial \psi}{\partial t} =& [a + |\widehat{\nabla} g|^2] \frac{\partial}{\partial t} |\widehat{\nabla}\widehat{\nabla} g|^2 + |\widehat{\nabla}\widehat{\nabla} g|^2 \frac{\partial}{\partial t}[a + |\widehat{\nabla} g|^2] \\ \le& [a + |\widehat{\nabla} g|^2] \Delta |\widehat{\nabla}\widehat{\nabla} g|^2 + |\widehat{\nabla}\widehat{\nabla} g|^2 \cdot \Delta[a + |\widehat{\nabla} g|^2] \\ & - [a + |\widehat{\nabla} g|^2] \cdot |\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g|^2 \\ & + C_{29} q(t)^{12} [a + |\widehat{\nabla} g|^2] \cdot |\widehat{\nabla}\widehat{\nabla} g|^3 \\ & + C_{30} q(t)^{30} [a + |\widehat{\nabla} g|^2] - |\widehat{\nabla}\widehat{\nabla} g|^4 \\ & + 4C_{18} q(t)^{12} |\widehat{\nabla}\widehat{\nabla} g|^2, \quad \tau_0 \le t \le T, \end{aligned} \tag{116}$$

$$\begin{aligned} \frac{\partial \psi}{\partial t} \le& [a + |\widehat{\nabla} g|^2] \Delta |\widehat{\nabla}\widehat{\nabla} g|^2 + |\widehat{\nabla}\widehat{\nabla} g|^2 \cdot \Delta[a + |\widehat{\nabla} g|^2] \\ & - [a + |\widehat{\nabla} g|^2] \cdot |\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g|^2 - \frac{1}{2} |\widehat{\nabla}\widehat{\nabla} g|^4 \\ & + C_{31} q(t)^{48} [a + |\widehat{\nabla} g|^2]^4 + C_{30} q(t)^{30} [a + |\widehat{\nabla} g|^2] \\ & + C_{32} q(t)^{24}, \quad \tau_0 \le t \le T, \end{aligned} \tag{117}$$

where $0 < C_{31}, C_{32} < +\infty$ are constants depending only on n, k_0, θ and C_1. On the other hand we have

$$\begin{aligned} [a+&|\widehat{\nabla} g|^2] \Delta |\widehat{\nabla}\widehat{\nabla} g|^2 + |\widehat{\nabla}\widehat{\nabla} g|^2 \Delta[a + |\widehat{\nabla} g|^2] \\ &= \Delta\psi - 2g^{ij} \nabla_j |\widehat{\nabla}\widehat{\nabla} g|^2 \cdot \nabla_i [a + |\widehat{\nabla} g|^2] \\ &= \Delta\psi - 2g^{ij} \widehat{\nabla}_j |\widehat{\nabla}\widehat{\nabla} g|^2 \cdot \widehat{\nabla}_i |\widehat{\nabla} g|^2 \\ &= \Delta\psi + g^{-1} * \widehat{\nabla}_i [g^{-3} * \widehat{\nabla} g * \widehat{\nabla} g] \\ &\qquad * \widehat{\nabla}_j [g^{-4} * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g] \\ &= \Delta\psi + g^{-1} * [g^{-3} * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g + g^{-4} * \widehat{\nabla} g * \widehat{\nabla} g * \widehat{\nabla} g] \\ &\qquad * [g^{-4} * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g + g^{-5} * \widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g * \widehat{\nabla}\widehat{\nabla} g], \end{aligned} \tag{118}$$

which together with (60), (62) and Lemma 7.4 yields

$$\begin{aligned}
&[a+|\widehat{\nabla} g|^2]\Delta|\widehat{\nabla}\widehat{\nabla} g|^2+|\widehat{\nabla}\widehat{\nabla} g|^2\Delta[a+|\widehat{\nabla} g|^2] \\
&\quad \leq \Delta\psi + C_{33}q(t)^{11}|\widehat{\nabla}\widehat{\nabla} g|^2\cdot|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g| \\
&\qquad + C_{34}q(t)^{15}|\widehat{\nabla}\widehat{\nabla} g|^3 + C_{35}q(t)^{18}|\widehat{\nabla}\widehat{\nabla} g|\cdot|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g| \\
&\qquad + C_{36}q(t)^{22}|\widehat{\nabla}\widehat{\nabla} g|^2 \\
&\quad \leq \Delta\psi + \frac{a}{2}|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g|^2 + \frac{C_{33}^2}{a}q(t)^{22}|\widehat{\nabla}\widehat{\nabla} g|^4 \\
&\qquad + \frac{C_{35}^2}{a}q(t)^{36}|\widehat{\nabla}\widehat{\nabla} g|^2 + C_{34}q(t)^{15}|\widehat{\nabla}\widehat{\nabla} g|^3 \\
&\qquad + C_{36}q(t)^{22}|\widehat{\nabla}\widehat{\nabla} g|^2 \\
&\quad \leq \Delta\psi + \frac{a}{2}|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g|^2 + \frac{C_{33}^2}{a}q(t)^{22}|\widehat{\nabla}\widehat{\nabla} g|^4 \\
&\qquad + \frac{1}{4}|\widehat{\nabla}\widehat{\nabla} g|^4 + \frac{8C_{35}^4}{a^2}q(t)^{72} + 8C_{34}^4q(t)^{60} \\
&\qquad + 8C_{36}^2q(t)^{44}, \quad \tau_0\leq t\leq T,
\end{aligned} \tag{119}$$

where $0<C_{33},\ C_{34},\ C_{35},\ C_{36}<+\infty$ are constants depending only on $n,\ k_0,\ \theta$ and C_1. Substituting (119) into (117) we get

$$\begin{aligned}
\frac{\partial\psi}{\partial t} \leq &\Delta\psi - \left[\frac{a}{2}+|\widehat{\nabla} g|^2\right]\cdot|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g|^2 \\
&-\left[\frac{1}{4}-\frac{C_{33}^2}{a}q(t)^{22}\right]\cdot|\widehat{\nabla}\widehat{\nabla} g|^4 + \frac{8C_{35}^4}{a^2}q(t)^{72} \\
&+8C_{34}^4q(t)^{60}+8C_{36}^2q(t)^{44}+C_{32}q(t)^{24} \\
&+C_{31}q(t)^{48}[a+|\widehat{\nabla} g|^2]^4 + C_{30}q(t)^{30}[a+|\widehat{\nabla} g|^2], \\
&\qquad\qquad\qquad\qquad \tau_0\leq t\leq T.
\end{aligned} \tag{120}$$

For any $t_0\in[\tau_0,T]$, from Lemma 7.4 we get

$$\sup_{x\in M}|\widehat{\nabla} g(x,t)|^2 \leq C_{16}q(t_0)^6, \quad \tau_0\leq t\leq t_0. \tag{121}$$

Now we choose a such that

$$a = 1+16(C_{16}+C_{33}^2)q(t_0)^{22}. \tag{122}$$

Then

$$a\leq a+|\widehat{\nabla} g(x,t)|^2\leq \frac{17}{16}a, \quad \text{on } M\times[\tau_0,t_0], \tag{123}$$

$$\frac{1}{4}-\frac{C_{33}^2}{a}q(t)^{22}\geq\frac{1}{8}, \quad \tau_0\leq t\leq t_0. \tag{124}$$

Combining (120), (122), (123) and (124) we obtain

$$\frac{\partial \psi}{\partial t} \leq \Delta\psi - \frac{a}{2}|\widehat{\nabla}\widehat{\nabla}\widehat{\nabla} g|^2 - \frac{1}{8}|\widehat{\nabla}\widehat{\nabla} g|^4 + C_{37} q(t_0)^{136}, \quad \tau_0 \leq t \leq t_0, \tag{125}$$

where $0 < C_{37} < +\infty$ is a constant depending only on n, k_0, θ and C_1. Since from (123) we have

$$|\widehat{\nabla}\widehat{\nabla} g|^4 = \frac{\psi^2}{[a + |\widehat{\nabla} g|^2]^2} \geq \frac{1}{4a^2}\psi^2, \quad \tau_0 \leq t \leq t_0, \tag{126}$$

which together with (125) implies

$$\frac{\partial \psi}{\partial t} \leq \Delta\psi - \frac{1}{32a^2}\psi^2 + C_{37} q(t_0)^{136}, \quad \tau_0 \leq t \leq t_0. \tag{127}$$

By the definition, $\psi(x, \tau_0) \equiv 0$ on M. Using Lemma 4.11 from (127) we get

$$\psi(x,t) \leq C_{38} q(t_0)^{68} a, \quad \tau_0 \leq t \leq t_0, \tag{128}$$

where $0 < C_{38} < +\infty$ is a constant depending only on n, k_0, θ and C_1. From (123) it follows that

$$\psi(x,t) \geq a|\widehat{\nabla}\widehat{\nabla} g|^2, \quad \tau_0 \leq t \leq t_0, \tag{129}$$

which together with (128) implies

$$\sup_{x\in M} |\widehat{\nabla}\widehat{\nabla} g(x,t)|^2 \leq C_{38} q(t_0)^{68}, \quad \tau_0 \leq t \leq t_0.$$

Let $t = t_0$. Then

$$\sup_{x\in M} |\widehat{\nabla}\widehat{\nabla} g(x,t_0)|^2 \leq C_{38} q(t_0)^{68}. \tag{130}$$

Since $t_0 \in [\tau_0, T]$ is arbitrary, we have

Lemma 7.5. *There exists a constant* $0 < C_{38} < +\infty$ *depending only on* n, k_0, θ *and* C_1 *such that*

$$\sup_{x\in M} |\widehat{\nabla}\widehat{\nabla} g(x,t)|^2 \leq C_{38} q(t)^{68}, \quad \tau_0 \leq t \leq T. \tag{131}$$

Combining (18) and (25) we know that

$$\begin{aligned} R_{\alpha\overline{\beta}} &= g^{-1} * \widehat{\nabla}\widehat{\nabla} g + g^{-2} * \widehat{\nabla} g * \widehat{\nabla} g + \widehat{R}_{\alpha\overline{\beta}} \\ &= g^{-1} * \widehat{\nabla}\widehat{\nabla} g + g^{-2} * \widehat{\nabla} g * \widehat{\nabla} g + \widehat{g}^{-1} * \widehat{Rm}, \end{aligned} \tag{132}$$

which together with (11), (15), (60), (62), Lemma 7.4 and Lemma 7.5 implies

$$0 \le R_{\alpha\overline{\beta}}(x,t) \le C_{39} q(t)^{35} \widehat{g}_{\alpha\overline{\beta}}(x), \quad \tau_0 \le t \le T, \tag{133}$$

$$\sup_{x\in M} |R_{\alpha\overline{\beta}}(x,t)|^2 \le C_{40} q(t)^{72}, \quad \tau_0 \le t \le T, \tag{134}$$

where $0 < C_{39},\ C_{40} < +\infty$ are constants depending only on $n,\ k_0,\ \theta$ and C_1. Combining (11) and (134) we get

$$\sup_{x\in M} |R_{ijkl}(x,t)|^2 \le C_{41} q(t)^{72}, \quad \tau_0 \le t \le T, \tag{135}$$

where $0 < C_{41} < +\infty$ is a constant depending only on $n,\ k_0,\ \theta$ and C_1. From (9) it follows that

$$\sup_{x\in M} |R_{ijkl}(x,t)|^2 \le C(n,0) k_0^2, \quad 0 \le t \le \tau_0, \tag{136}$$

which together with (135) yields

Lemma 7.6. *Under Assumption E, there exists a constant $0 < C_{42} < +\infty$ depending only on $n,\ k_0,\ \theta$ and C_1 such that*

$$\sup_{x\in M} |R_{ijkl}(x,t)|^2 \le C_{42} q(t)^{72}, \quad 0 \le t \le T. \tag{137}$$

Now we want to estimate the covariant derivatives of the curvature tensor. For any $t_0 \in (0,T]$, from (137) we know that

$$\sup_{M\times[\frac{t_0}{2},t_0]} |R_{ijkl}(x,t)|^2 \le C_{42} q(t_0)^{72}, \tag{138}$$

which together with Lemma 2.3 implies

$$\begin{aligned} &\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \\ &\le C(n,m)\cdot C_{42} q(t_0)^{72} \left\{ \left(\frac{1}{t-\frac{1}{2}t_0} \right)^m + C_{42}^{\frac{m}{2}} q(t_0)^{36m} \right\}, \\ &\qquad \frac{t_0}{2} \le t \le t_0, \end{aligned} \tag{139}$$

where $m \geq 0$ is any integer, and $C(n,m)$ are constants depending only on n and m. Let $t = t_0$. Then by (139) we get

$$\sup_{x\in M}|\nabla^m R_{ijkl}(x,t_0)|^2 \leq C(n,m)\cdot C_{42}q(t_0)^{72}\left\{\frac{2^m}{t_0^m}+C_{42}^{\frac{m}{2}}q(t_0)^{36m}\right\}. \tag{140}$$

Since $t_0 \in (0,T]$ is arbitrary, from (140) follows

Lemma 7.7. *Under Assumption E, for any integers $m \geq 0$, there exist constants $0 < C_{43}(m,\ n,\ k_0,\ \theta, C_1) < +\infty$ depending only on $m,\ n,\ k_0,\ \theta$ and C_1 such that*

$$\sup_{x\in M}|\nabla^m R_{ijkl}(x,t)|^2 \leq C_{43}(m,n,k_0,\theta,C_1)\left\{\left(\frac{1}{t}\right)^m + q(t)^{36(m+2)}\right\}, \quad 0 \leq t \leq T. \tag{141}$$

Let $t = T$. Then by Lemma 7.6 we obtain

$$\sup_{x\in M}|R_{ijkl}(x,T)|^2 \leq C_{42}q(T)^{72}. \tag{142}$$

Suppose $\theta_0(n)$ is the constant in Corollary 2.2, we define

$$T_1 = T + \frac{\theta_0(n)}{\sqrt{C_{42}}q(T)^{36}}. \tag{143}$$

Then from (142), Corollary 2.2 and Lemma 7.7 it follows that the solution $g_{ij}(x,t)$ of (1) on $M\times[0,T]$ can be extended smoothly to a solution $g_{ij}(x,t)$ of (1) on $M\times[0,T_1]$ satisfying

$$\sup_{x\in M}|R_{ijkl}(x,t)|^2 \leq C(n,0)\cdot C_{42}q(T)^{72}, \quad T \leq t \leq T_1, \tag{144}$$

where $C(n,0)$ is the constant in Corollary 2.2. (144) together with Lemma 7.6 yields

$$\sup_{M\times[0,T_1]}|R_{ijkl}(x,t)|^2 \leq [1+C(n,0)]\cdot C_{42}q(T)^{72}. \tag{145}$$

Therefore we have

Lemma 7.8. *Suppose $g_{ij}(x,t) > 0$ is a smooth solution of the Ricci flow equation (1) on $M \times [0,T]$ such that Assumption E holds on $M \times [0,T]$. Then $g_{ij}(x,t)$ can be extended smoothly to a solution of the Ricci flow equation (1) on $M \times [0,T_1]$ such that Assumption E still holds on $M \times [0,T_1]$.*

Under Assumption D, by Lemma 7.1 there exists a constant $0 < T < +\infty$ such that the Ricci flow equation (1) has a smooth solution $g_{ij}(x,t) > 0$ on $M \times [0,T]$ and Assumption E holds on $M \times [0,T]$. Now using Lemma 7.8 repeatedly we know that $g_{ij}(x,t)$ can be extended smoothly to a solution to the Ricci flow equation (1) on $M \times [0,\infty)$ such that for any $T_0 \in [T,\infty)$, Assumption E still hold on $M \times [0,T_0]$. Hence

Theorem 7.9. *Under Assumption D, there exists a smooth solution $g_{ij}(x,t) > 0$ to the Ricci flow equation (1) on $M \times [0,\infty)$ such that for any $T_0 \in [\tau_0,\infty)$, Assumption E hold on $M \times [0,T_0]$.*

Since for any $T_0 \in [\tau_0,\infty)$, Assumption E holds on $M \times [0,T_0]$, combining Lemma 7.2, Lemma 7.6 and Lemma 7.7 we get

Theorem 7.10. *Under Assumption D, there exists a smooth solution $g_{ij}(x,t) > 0$ to the Ricci flow equation (1) on $M \times [0,\infty)$ such that*

(A) $g_{ij}(x,t)$ *are Kähler metrics for any* $0 \le t < +\infty$,

(B) $-R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) \ge 0, \quad$ *on* $M \times [0,\infty)$,

(C) $F(x,t) \ge -C_2(t+2)^{\frac{2-\theta}{\theta}}, \quad$ *on* $M \times [0,\infty)$,

(D) $ds_0^2 \ge ds_t^2 \ge \dfrac{1}{q(t)}\, ds_0^2, \quad 0 \le t < +\infty$,

(E) $\sup\limits_{x\in M} |R_{ijkl}(x,t)|^2 \le C_{42} q(t)^{72}, \quad 0 \le t < +\infty$,

(F) $$\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \le C_{43}(m)\left\{\left(\frac{1}{t}\right)^m + q(t)^{36(m+2)}\right\},$$
$$m \ge 0,\ 0 \le t < +\infty.$$

8. Controlling the curvature tensor

Suppose $g_{ij}(x,t) > 0$ is the smooth solution on $M \times [0,\infty)$ of the

Ricci flow equation

$$(1)\qquad \begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), & \text{on } M \times [0,\infty), \\ g_{ij}(x,0) = \widetilde{g}_{ij}(x), & \text{on } M, \end{cases}$$

which we obtained in Theorem 7.10. In Theorem 7.10 (E) we also obtained some estimate for the curvature tensor $\{R_{ijkl}(x,t)\}$. However, that estimate is not optimal. To prove our main result Theorem 1.2, we need a better estimate for the curvature tensor $\{R_{ijkl}(x,t)\}$ than the estimate obtained in Theorem 7.10 (E). Under the assumptions of Theorem 1.1, the author of this paper derived a complicated integral estimate technique to improve the curvature tensor estimate obtained in Theorem 7.10 in his Ph.D. thesis [43] in 1990. Later on H.D. Cao [10] and R.S. Hamilton [24] proved the Harnack's inequality for the Ricci flow equation in 1992. The consequence of their results improves the curvature tensor estimate which we obtained in Theorem 7.10 (E). In this section we use the noncompact version of their results.

Theorem 8.1. *Suppose that $g_{ij}(x,t) > 0$ is the smooth solution to the Ricci flow equation (1) on $M \times [0,\infty)$ which we obtained in Theorem 7.10, and that the holomorphic bisectional curvature of $g_{ij}(x,t)$ is strictly positive, i.e.,*

$$(2)\qquad -R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) > 0, \quad \textit{on } M \times [0,\infty).$$

Then the scalar curvature $R(x,t)$ of $g_{ij}(x,t)$ satisfies the inequality:

$$(3)\qquad \frac{\partial R}{\partial t} - 2\frac{|\nabla_\alpha R|^2}{R} + \frac{1}{t}R > 0, \quad \textit{on } M \times (0,\infty).$$

Proof. Suppose M is a compact Kähler manifold, and $g_{\alpha\overline{\beta}}(x,t) > 0$ is a smooth family of Kähler metrics on M such that

$$(4)\qquad \frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x,t) = -2R_{\alpha\overline{\beta}}(x,t), \quad \text{on } M \times [0,T],$$

where $0 < T < +\infty$ is a constant. Suppose that the holomorphic bisectional curvature of $g_{\alpha\overline{\beta}}(x,t)$ is strictly positive:

$$(5)\qquad -R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) > 0, \quad \text{on } M \times [0,T].$$

We now define

$$(6)\qquad \widehat{g}_{\alpha\overline{\beta}}(x,s) = \frac{1}{2T} e^s g_{\alpha\overline{\beta}}(x, T(1-e^{-s})), \quad \text{on } M \times [0,\infty).$$

Then $\widehat{g}_{\alpha\overline{\beta}}(x,s) > 0$ is also a smooth family of Kähler metrics on M, which satisfies the normalized Ricci flow equation:

$$\frac{\partial}{\partial s}\widehat{g}_{\alpha\overline{\beta}}(x,s) = -\widehat{R}_{\alpha\overline{\beta}}(x,s) + \widehat{g}_{\alpha\overline{\beta}}(x,s), \quad \text{on } M \times [0,\infty), \tag{7}$$

where $\widehat{R}_{\alpha\overline{\beta}}(x,s)$ denotes the Ricci curvature of $\widehat{g}_{\alpha\overline{\beta}}(x,s)$. It is easy to see that the holomorphic bisectional curvature of $\widehat{g}_{\alpha\overline{\beta}}(x,s)$ is also strictly positive, i.e.,

$$-\widehat{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,s) > 0, \quad \text{on } M \times [0,\infty). \tag{8}$$

From Corollary 4.1 in H.D. Cao [10] it follows that the scalar curvature $\widehat{R}(x,s)$ of $\widehat{g}_{\alpha\overline{\beta}}(x,s)$ satisfies the inequality:

$$\frac{\partial \widehat{R}}{\partial s} - \frac{|\widehat{\nabla}_\alpha \widehat{R}|^2}{\widehat{R}} + \frac{\widehat{R}}{1-e^{-s}} > 0, \quad \text{on } M \times (0,\infty), \tag{9}$$

where $\widehat{\nabla}$ denote the covariant derivatives with respect to $\widehat{g}_{\alpha\overline{\beta}}(x,s)$. Combining (6) and (9) implies that the scalar curvature $R(x,t)$ of $g_{\alpha\overline{\beta}}(x,t)$ satisfies the inequality:

$$\frac{\partial R}{\partial t} - 2\frac{|\nabla_\alpha R|^2}{R} + \frac{1}{t}R > 0, \quad \text{on } M \times (0,T), \tag{10}$$

where ∇ denote the covariant derivatives with respect to $g_{\alpha\overline{\beta}}(x,t)$.

Now suppose $g_{ij}(x,t) > 0$ is the smooth solution to the Ricci flow equation (1) on $M\times[0,\infty)$, which we obtained in Theorem 7.10, and suppose assumption (2) in Theorem 8.1 holds. For any constant $0 < T < +\infty$, from Theorem 7.10 (E) we know that the curvature tensors of $g_{ij}(x,t)$ are uniformly bounded on $M \times [0,T]$:

$$\sup_{M\times[0,T]} |R_{ijkl}(x,t)|^2 \leq \Theta, \tag{11}$$

where $0 < \Theta < +\infty$ is a constant depending only on T and the constants n, k_0, θ and C_1 in Assumption D in §7. Since under Assumption D, the manifold M is complete and noncompact, we have to try to control the curvature of $g_{ij}(x,t)$ and the other tensors near the infinity of M if we want to use the method in the paper of Cao [10] to prove that the scalar curvature $R(x,t)$ of $g_{ij}(x,t)$ still satisfies the inequality (10) on $M \times (0,T)$. In his paper [24] R.S. Hamilton derived some kind of cut–off function technique which was used to control the curvature and the other tensors near the infinity of the manifold when Hamilton proved the Harnack estimate for the Ricci flow equation on complete noncompact manifolds with bounded and positive curvature operator. From (2) and (11) it is easy to see that the cut–off function technique in the paper of Hamilton [24] can still be used in our case. Thus combining the techniques in [10] and [24] we know that the scalar curvature $R(x,t)$ of $g_{ij}(x,t)$ in Theorem 8.1 satisfies the inequality (10) on $M \times (0,T)$. Since $T \in (0,\infty)$ is arbitrary, we know that (3) is true on $M \times (0,\infty)$.

Now suppose $g_{ij}(x,t) > 0$ is the smooth solution on $M \times [0,\infty)$ to the Ricci flow equation (1) which we obtained in Theorem 7.10. We want to use Theorem 8.1 to improve the curvature tensor estimate obtained in Theorem 7.10. We assume that the holomorphic bisectional curvature of $g_{ij}(x,t)$ is strictly positive:

$$-R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) > 0, \quad \text{on } M \times [0,\infty). \tag{12}$$

Suppose $F(x,t)$ is the function defined by (9) of §6. From (11) and (14) of §6 it follows that

$$\begin{aligned} F(x,t) &= F(x,0) + \int_0^t \frac{\partial}{\partial s} F(x,s)ds \\ &= -\int_0^t R(x,s)ds, \quad \text{on } M \times [0,\infty), \end{aligned} \tag{13}$$

which together with Theorem 7.10 (C) implies

$$\int_0^t R(x,s)ds \le C_2(t+2)^{\frac{2-\theta}{\theta}}, \quad \text{on } M \times [0,\infty), \tag{14}$$

where $0 < C_2 < +\infty$ is a constant depending only on n, k_0, θ and C_1. By (12) we have

$$R(x,t) > 0, \quad \text{on } M \times [0,\infty). \tag{15}$$

From Theorem 8.1 we know that $R(x,t)$ satisfies the inequality:

$$\frac{\partial R}{\partial t} > 2\frac{|\nabla_\alpha R|^2}{R} - \frac{1}{t}R, \quad \text{on } M \times (0,\infty), \tag{16}$$

which together with (15) yields

$$\frac{\partial R}{\partial t} > -\frac{1}{t}R, \quad \text{on } M \times (0,\infty). \tag{17}$$

Combining (14), (15) and (17) we get

$$0 < R(x,t) \le C_3(t+2)^{\frac{2-2\theta}{\theta}}, \quad \text{on } M \times [0,\infty), \tag{18}$$

where $0 < C_3 < +\infty$ is a constant depending only on n, k_0, θ and C_1. From (12) it follows that there exists a constant $0 < C_4 < +\infty$ depending only on n such that

$$|R_{ijkl}(x,t)| \le C_4 R(x,t), \quad \text{on } M \times [0,\infty). \tag{19}$$

Combining (18) and (19) thus leads to

Theorem 8.2. *Suppose that $g_{ij}(x,t) > 0$ is the smooth solution to the Ricci flow equation (1) on $M \times [0,\infty)$, which we obtained in Theorem 7.10, and that the holomorphic bisectional curvature of $g_{ij}(x,t)$ is strictly positive:*

$$-R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) > 0, \quad \text{on } M \times [0,\infty). \tag{20}$$

Then there exists a constant $0 < C_5 < +\infty$ depending only on n, k_0, θ and C_1 such that

$$\sup_{x\in M} |R_{ijkl}(x,t)| \le C_5(t+2)^{\frac{2-2\theta}{\theta}}, \quad 0 \le t < +\infty. \tag{21}$$

If we replace Lemma 7.6 by Theorem 8.2, then by the same technique as the technique used in the proof of Lemma 7.7 we get

Theorem 8.3. *Suppose that $g_{ij}(x,t) > 0$ is the smooth solution to the Ricci flow equation (1) on $M \times [0,\infty)$, which we obtained in Theorem 7.10, and that the holomorphic bisectional curvature of $g_{ij}(x,t)$ is strictly positive:*

$$-R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) > 0, \quad \text{on } M \times [0,\infty). \tag{22}$$

Then for any integers $m \ge 0$, there exist constants

$$0 < C_6(m,\ n,\ k_0,\ \theta,\ C_1) < +\infty$$

depending only on m, n, k_0, θ and C_1 such that

$$\begin{aligned} &\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \\ &\le C_6(m,\ n,\ k_0,\ \theta,\ C_1) \cdot \frac{(t+2)^{(\frac{2-\theta}{\theta})(m+2)}}{(t+2)^2 t^m}, \\ &\qquad 0 \le t < +\infty. \end{aligned} \tag{23}$$

Remark. In Assumption D in §7, we assumed that $0 < \theta < 2$. If the constant $\theta = 2$ in Assumption D, using (145) of §6 and the same method as that used in the proof of Theorem 8.3 we know that (23) is replaced by

$$\begin{aligned} &\sup_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \\ &\leq C_6(m,\ n,\ k_0,\ C_1)\cdot\frac{[\log(t+2)]^{m+2}}{(t+2)^2 t^m}, \quad 0 \leq t < +\infty. \end{aligned} \tag{24}$$

Under the assumptions of Theorem 1.1, a result similar to (24) was proved by the author of this paper in [43] in 1990.

9. Constructing the biholomorphic maps

In this section we always assume that the assumptions in Theorem 1.2 hold. Suppose

$$d\widetilde{s}^2 = \widetilde{g}_{ij}(x)dx^i dx^j > 0 \tag{1}$$

is the complete Kähler metric on M with bounded and positive sectional curvature:

$$0 < \widetilde{R}_{ijij}(x) \leq k_0, \quad \forall\ x \in M, \tag{2}$$

and satisfies

$$\begin{aligned} \textstyle\int_{B(x_0,\gamma)} \widetilde{R}(x)dx \leq \frac{C_1}{(\gamma+1)^{1+\varepsilon}} \cdot \mathrm{Vol}(B(x_0,\gamma)), \quad x_0 \in M, \\ 0 \leq \gamma < +\infty, \end{aligned} \tag{3}$$

where $0 < k_0,\ C_1 < +\infty,\ 0 < \varepsilon < 1$ are constants. From (2) it follows that the holomorphic bisectional curvature of $d\widetilde{s}^2$ is bounded and positive:

$$0 < -\widetilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}}(x) \leq 2k_0, \quad \forall\ x \in M. \tag{4}$$

By the assumptions in Theorem 1.2, $(M, \widetilde{g}_{ij}(x))$ is a complex n–dimensional complete noncompact Kähler manifold. Thus Assumption D in §7 holds with the constant $\theta = 1 + \varepsilon$.

Lemma 9.1. *There exists a smooth solution $g_{ij}(x,t) > 0$ to the Ricci flow equation*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), & \text{on } M \times [0,\infty), \\ g_{ij}(x,0) = \widetilde{g}_{ij}(x), & \text{on } M, \end{cases} \tag{5}$$

on $M \times [0,\infty)$ such that

(A) $g_{ij}(x,t)$ *are Kähler metrics for any* $0 \le t < +\infty$,

(B) $-R_{\alpha\overline{\alpha}\beta\overline{\beta}}(x,t) > 0$, *on* $M \times [0,\infty)$,

(C) $F(x,t) \ge -C_2(t+2)^{\frac{1-\varepsilon}{1+\varepsilon}}$, *on* $M \times [0,\infty)$,

(D) $ds_0^2 \ge ds_t^2 \ge \dfrac{1}{q(t)}\, ds_0^2$, $0 \le t < +\infty$,

(E) $\sup\limits_{x\in M} |R_{ijkl}(x,t)| \le C_3(t+2)^{-\frac{2\varepsilon}{1+\varepsilon}}$, $0 \le t < +\infty$,

(F) $\sup\limits_{x\in M} |\nabla^m R_{ijkl}(x,t)|^2 \le C_4(m) \cdot \dfrac{(t+2)^{(\frac{1-\varepsilon}{1+\varepsilon})(m+2)}}{(t+2)^2 t^m}$, $m \ge 0$, $0 \le t < +\infty$,

where $0 < C_2$, $C_3 < +\infty$ are constants depending only on n, k_0, ε and C_1, $0 < C_4(m) < +\infty$ are constants depending only on m, n, k_0, ε and C_1, $F(x,t)$ is defined by (9) in §6,

$$ds_t^2 = g_{ij}(x,t)dx^i dx^j, \quad 0 \le t < +\infty, \tag{6}$$

$$q(t) = e^{C_2(t+2)^{\frac{1-\varepsilon}{1+\varepsilon}}}, \quad 0 \le t < +\infty. \tag{7}$$

Proof. From Theorem 7.10 we know that there exists a smooth solution $g_{ij}(x,t) > 0$ to the Ricci flow equation (5) on $M \times [0,\infty)$ such that (A), (C) and (D) of Lemma 9.1 hold. Combining (4), Theorem 7.10 (E) and Theorem 5.5 yields that Lemma 9.1 (B) holds. From Theorem 8.2 and Theorem 8.3 it follows that (E) and (F) of Lemma 9.1 hold.

q.e.d.

For any two points x, $y \in M$, let $\gamma_t(x,y)$ denote the distance between x and y with respect to ds_t^2. Let $B_t(x,\gamma)$ denote the geodesic ball of radius γ and centered at $x \in M$ with respect to ds_t^2.

Now we fix a point $x_0 \in M$. We use $T_{x_0}M$ to denote the space of all the holomorphic tangent vectors of M at x_0. For any two holomorphic vectors $V_1, V_2 \in T_{x_0}M$, let $\langle V_1, V_2\rangle_t$ denote the inner product of V_1 and V_2 with respect to the metric ds_t^2. Since $\{ds_t^2|0 \le t < +\infty\}$ is a family of Kähler metrics on M, which depends on t smoothly, it is easy to see that we can find $V_1(t),\ V_2(t), \dots, V_n(t) \in T_{x_0}M$ for $0 \le t < +\infty$ such that $V_1(t), V_2(t), \dots, V_n(t)$ depend on t smoothly and satisfy

$$\langle V_\alpha(t), V_\beta(t)\rangle_t \equiv \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \dots, n,\ 0 \le t < +\infty. \tag{8}$$

Thus

$$\begin{aligned} T_{x_0}M &= \bigoplus_{\alpha=1}^{n} \mathbb{C}V_\alpha(t) \\ &= \left\{ \sum_{\alpha=1}^{n} z^\alpha V_\alpha(t) | z^1, z^2, \dots, z^n \in \mathbb{C} \right\}, \quad 0 \le t < +\infty. \end{aligned} \tag{9}$$

For each $t \in [0, \infty)$, we define a linear map ψ_t:

$$\begin{aligned} &\psi_t\colon T_{x_0}M \to \mathbb{C}^n, \\ &\psi_t\left(\sum_{\alpha=1}^{n} z^\alpha V_\alpha(t)\right) = (z^1, z^2, \dots, z^n),\ \forall\ z^1, z^2, \dots, z^n \in \mathbb{C}. \end{aligned} \tag{10}$$

Then $\{\psi_t|0 \le t < +\infty\}$ is a family of invertible linear maps between $T_{x_0}M$ and $\mathbb{C}^n$, which depends on t smoothly. For each $t \in [0, \infty)$, we use

$$\exp_{x_0}^t\colon T_{x_0}M \to M \tag{11}$$

to denote the exponential map with respect to the metric ds_t^2. We now define maps

$$\begin{aligned} &\Psi_t\colon \mathbb{C}^n \to M, \\ &\Psi_t = \exp_{x_0}^t \circ \psi_t^{-1}, \quad 0 \le t < +\infty. \end{aligned} \tag{12}$$

Then $\{\Psi_t|0 \le t < +\infty\}$ is a family of smooth maps from $\mathbb{C}^n$ to M, which depends on t smoothly and satisfies

$$\Psi_t(0) = x_0, \quad 0 \le t < +\infty. \tag{13}$$

Suppose

$$\mathbb{C}^n = \{z = (z^1, z^2, \dots, z^n)| z^1, z^2, \dots, z^n \in \mathbb{C}\}, \tag{14}$$

and use

$$d\hat{s}^2 = \sum_{\alpha=1}^{n} dz^\alpha d\overline{z}^\alpha \tag{15}$$

to denote the standard flat Kähler metric on $\mathbb{C}^n$. For any $z \in \mathbb{C}^n$ and $\gamma > 0$, let

$$\widehat{B}(z,\gamma) = \{w \in \mathbb{C}^n \,||w - z| < \gamma\} \tag{16}$$

denote the geodesic ball of radius γ and centered at z with respect to $d\hat{s}^2$. Let $\widehat{\nabla}$ denote the covariant derivatives with respect to the metric $d\hat{s}^2$. From Lemma 9.1 it follows that

$$\begin{aligned} \sup_{x\in M} |\nabla^m R_{ijkl}(x,t)| \le C_5(m)\cdot(t+2)^{-\frac{\varepsilon}{1+\varepsilon}(m+2)}, \\ m \ge 0,\ 2 \le t < +\infty, \end{aligned} \tag{17}$$

where $0 < C_5(m) < +\infty$ are constants depending only on n, m, k_0, ε and C_1. Thus the curvature tensor $\{R_{ijkl}(x,t)\}$ together with its covariant derivatives tend to zero uniformly on M as time $t \to +\infty$. We now define

$$\mathcal{U}_0(t) = (t+2)^{\frac{\varepsilon}{1+\varepsilon}}, \quad 0 \le t < +\infty. \tag{18}$$

Then by (17) there exists a constant $0 < C_6 \le 1$ depending only on n, k_0, ε and C_1 such that for any $t \in [2,\infty)$, the map Ψ_t is nonsingular on $\widehat{B}(0, C_6\mathcal{U}_0(t))$. Thus we consider the pull–back metric

$$\Psi_t^*(ds_t^2) = g^*_{AB}(z,t)dz^A dz^B, \quad z \in \widehat{B}(0, C_6\mathcal{U}_0(t)), \tag{19}$$

where $A, B = \alpha$ or $\overline{\alpha}$. Since Ψ_t are not holomorphic maps in general, the metrics in (19) are not Kähler with respect to z in general. However, by the definition of Ψ_t in (12), $g^*_{AB}(z,t)$ depend on t and z smoothly. From (12) and (17) it follows that there exists another constant C_7, $0 < C_7 \le C_6 \le 1$, depending only on n, k_0, ε and C_1, such that for any $t \in [2,\infty)$,

$$\begin{cases} |g^*_{\alpha\overline{\beta}}(z,t) - \delta_{\alpha\beta}| \le \frac{C_8|z|^2}{\mathcal{U}_0(t)^2}, & z \in \widehat{B}(0, C_7\mathcal{U}_0(t)), \\ |g^*_{\overline{\alpha}\beta}(z,t) - \delta_{\alpha\beta}| \le \frac{C_8|z|^2}{\mathcal{U}_0(t)^2}, & z \in \widehat{B}(0, C_7\mathcal{U}_0(t)), \\ |g^*_{\alpha\beta}(z,t)| \le \frac{C_8|z|^2}{\mathcal{U}_0(t)^2}, & z \in \widehat{B}(0, C_7\mathcal{U}_0(t)), \\ |g^*_{\overline{\alpha}\overline{\beta}}(z,t)| \le \frac{C_8|z|^2}{\mathcal{U}_0(t)^2}, & z \in \widehat{B}(0, C_7\mathcal{U}_0(t)), \end{cases} \tag{20}$$

$$|\widehat{\nabla} g^*_{AB}(z,t)| \le \frac{C_8|z|}{\mathcal{U}_0(t)^2}, \quad z \in \widehat{B}(0, C_7\mathcal{U}_0(t)), \tag{21}$$

$$|\widehat{\nabla}\widehat{\nabla} g^*_{AB}(z,t)| \le \frac{C_8}{\mathcal{U}_0(t)^2}, \quad z \in \widehat{B}(0, C_7\mathcal{U}_0(t)), \tag{22}$$

where $0 < C_8 < +\infty$ is a constant depending only on n, k_0, ε and C_1. Let J denote the complex structure on M. Then $\Psi_t^*(J)$ defines a complex structure on $\widehat{B}(0, C_6\mathcal{U}_0(t))$, and $\Psi_t^*(ds_t^2)$ is a Kähler metric on $\widehat{B}(0, C_6\mathcal{U}_0(t))$ with respect to the complex structure $\Psi_t^*(J)$ for every $t \in [2, \infty)$. Suppose $\overline{\partial}$ is the $\overline{\partial}$–operator on M. For any $t \in [2, \infty)$, let $\overline{\partial}^t$ denote the $\overline{\partial}$–operator on $\widehat{B}(0, C_6\mathcal{U}_0(t))$ with respect to the complex structure $\Psi_t^*(J)$. It is easy to see that

$$\overline{\partial} = \Psi_t^*(\overline{\partial}^t), \quad 2 \le t < +\infty. \tag{23}$$

Define n holomorphic functions $p^1(z),\ p^2(z), \ldots, p^n(z)$ on $\mathbb{C}^n$:

$$p^\alpha(z) = z^\alpha,\ \forall\ z = (z^1, z^2, \ldots, z^n) \in \mathbb{C}^n,\ \alpha = 1, 2, \ldots, n. \tag{24}$$

Then by (20) and (23) we have

$$|\overline{\partial}^t p^\alpha(z)| \le \frac{C_9|z|^2}{\mathcal{U}_0(t)^2},\ z \in \widehat{B}(0, C_7\mathcal{U}_0(t)),\ \alpha = 1, 2, \ldots, n, \tag{25}$$

where $0 < C_9 < +\infty$ is a constant depending only on n, k_0, ε and C_1. We define

$$\mathcal{U}(t) = (t+2)^{\frac{\varepsilon}{2(1+\varepsilon)}}, \quad 0 \le t < +\infty. \tag{26}$$

Then

$$1 < \mathcal{U}(t) = \mathcal{U}_0(t)^{\frac{1}{2}} < \mathcal{U}_0(t), \quad 0 \le t < +\infty. \tag{27}$$

Combining (20), (21), (22), (25) and (27) we get

$$|g^*_{AB}(z,t) - \delta_{A\overline{B}}| \le \frac{C_{10}}{\mathcal{U}(t)^2}, \quad z \in \widehat{B}(0, C_7\mathcal{U}(t)), \tag{28}$$

$$|\widehat{\nabla} g^*_{AB}(z,t)| \le \frac{C_{10}}{\mathcal{U}(t)^3}, \quad z \in \widehat{B}(0, C_7\mathcal{U}(t)), \tag{29}$$

$$|\widehat{\nabla}\widehat{\nabla} g^*_{AB}(z,t)| \le \frac{C_{10}}{\mathcal{U}(t)^4}, \quad z \in \widehat{B}(0, C_7\mathcal{U}(t)), \tag{30}$$

$$|\overline{\partial}^t p^\alpha(z)| \le \frac{C_{10}}{\mathcal{U}(t)^2}, \quad z \in \widehat{B}(0, C_7\mathcal{U}(t)), \quad \alpha = 1, 2, \ldots, n, \tag{31}$$

and where $0 < C_{10} < +\infty$ is a constant depending only on n, k_0, ε and C_1,

$$\delta_{A\overline{B}} = \begin{cases} 1, & \text{if } A = \overline{B}, \\ 0, & \text{if } A \ne \overline{B}. \end{cases} \tag{32}$$

Let $\lambda(\gamma, t)$ denote the eigenvalues of the second fundamental form of $\partial\widehat{B}(0,\gamma)$ with respect to the metric $\Psi_t^*(ds_t^2)$. From (17), (28), (29) and (30) it follows that there exist constants

$$0 < C_{11}, C_{12}, C_{13} < +\infty$$

depending only on n, k_0, ε and C_1 such that

$$\frac{C_{12}}{\gamma} \le \lambda(\gamma, t) \le \frac{C_{13}}{\gamma}, \quad 0 < \gamma \le C_7\mathcal{U}(t),\ 2 + C_{11} \le t < +\infty. \tag{33}$$

Thus for any $t \ge 2 + C_{11}$ and $0 < \gamma \le C_7\mathcal{U}(t)$, $\partial\widehat{B}(0,\gamma)$ is convex with respect to the metric $\Psi_t^*(ds_t^2)$. For any $t \ge 2 + C_{11}$, using L^2 estimate theory for the $\overline{\partial}$ operator which appeared in the book of L. Hörmander [25], from (17), (28), (29), (30) and (31) we know that the $\overline{\partial}$ equations

$$\overline{\partial}^t\theta^\alpha(z,t) = \overline{\partial}^t p^\alpha(z),\ z \in \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)),\ \alpha = 1, 2, \dots, n, \tag{34}$$

have smooth solutions $\{\theta^\alpha(z,t) | \alpha = 1, 2, \dots, n\}$ such that

$$\begin{cases} |\theta^\alpha(z,t)| \le \frac{C_{14}}{\mathcal{U}(t)}, \\ |\widehat{\nabla}\theta^\alpha(z,t)| \le \frac{C_{14}}{\mathcal{U}(t)^2}, \end{cases} \quad z \in \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)), \quad \alpha = 1, 2, \dots, n, \tag{35}$$

where $0 < C_{14} < +\infty$ is a constant depending only on n, k_0, ε and C_1. If we choose the solutions $\theta^\alpha(z,t)$ of (34) such that $\theta^\alpha(z,t)$ are orthogonal to ker $\overline{\partial}^t$ in certain L^2 Hilbert function spaces on $\widehat{B}(0, C_7\mathcal{U}(t))$, and choose those L^2 function spaces such that those spaces depend on t smoothly, then $\theta^\alpha(z,t)$ depend on t smoothly. For how to choose those L^2 function spaces, one can see L. Hörmander [25] for details. Since $\mathcal{U}(t) \to +\infty$ as $t \to +\infty$, we can find a constant $2 + C_{11} \le C_{15} < +\infty$ depending only on n, k_0, ε and C_1 such that for any $t \ge C_{15}$ we have

$$C_7\mathcal{U}(t) \ge 2, \tag{36}$$

$$\begin{cases} |\theta^\alpha(z,t)| \le \frac{1}{2n}, \\ |\widehat{\nabla}\theta^\alpha(z,t)| \le \frac{1}{8n}, \end{cases} \quad z \in \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)),\ \alpha = 1, 2, \dots, n. \tag{37}$$

We now define a map $\widehat{\Psi}_t = (\widehat{\Psi}_t^1, \widehat{\Psi}_t^2, \dots, \widehat{\Psi}_t^n)$:

$$\begin{aligned} &\widehat{\Psi}_t : \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)) \to \mathbb{C}^n, \\ &\widehat{\Psi}_t^\alpha(z) = p^\alpha(z) - \theta^\alpha(z,t),\ z \in \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)),\ \alpha = 1, 2, \dots, n. \end{aligned} \tag{38}$$

From (34), (36) and (37) it follows that for any $t \geq C_{15}$, the map $\widehat{\Psi}_t$ is nonsingular on $\widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t))$ and satisfies

$$|\widehat{\Psi}_t(z) - z| \leq \frac{1}{2}, \ z \in \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)), \tag{39}$$

$$\widehat{\Psi}_t(\widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t))) \subset \widehat{B}(0, C_7\mathcal{U}(t)), \tag{40}$$

$$\overline{\partial}^t \widehat{\Psi}_t^\alpha(z) \equiv 0, \ z \in \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)). \tag{41}$$

By the definition, $\widehat{\Psi}_t$ depend on t smoothly. We let

$$\Phi_t = \Psi_t \circ \widehat{\Psi}_t \colon \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)) \to M. \tag{42}$$

Since $\widehat{\Psi}_t$ is nonsingular on $\widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t))$ and Ψ_t is nonsingular on $\widehat{B}(0, C_7\mathcal{U}(t))$, from (40) and (41) we know that $\{\Phi_t | C_{15} \leq t < +\infty\}$ is a family of nonsingular (nondegenerate) holomorphic maps from $\widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t))$ to M, which depends on t smoothly. From (13), (28), (35) and (39) it follows that for any $t \geq C_{15}$,

$$\gamma_t(x_0, \Phi_t(0)) \leq \frac{1}{2}\left(1 + \frac{C_{10}}{\mathcal{U}(t)^2}\right), \tag{43}$$

$$\left(1 - \frac{C_{16}}{\mathcal{U}(t)^2}\right) d\widehat{s}^2 \leq \Phi_t^*(ds_t^2) \leq \left(1 + \frac{C_{16}}{\mathcal{U}(t)^2}\right) d\widehat{s}^2, \quad \text{on } \widehat{B}(0, \frac{1}{2}C_7\mathcal{U}(t)), \tag{44}$$

where $0 < C_{16} < +\infty$ is a constant depending only on n, k_0, ε and C_1. Since $\mathcal{U}(t) \to +\infty$ as $t \to +\infty$, we get

Lemma 9.2. *There exist constants $0 < C_{17}, C_{18} < +\infty$ depending only on n, k_0, ε and C_1 such that $\{\Phi_t | C_{17} \leq t < +\infty\}$ is a family of nonsingular (nondegenerate) holomorphic maps from $\widehat{B}(0, C_{18}\mathcal{U}(t))$ to M, which depends on t smoothly and satisfies*

$$\gamma_t(x_0, \Phi_t(0)) < 1, \quad C_{17} \leq t < +\infty, \tag{45}$$

$$\frac{1}{2}d\widehat{s}^2 \leq \Phi_t^*(ds_t^2) \leq 2d\widehat{s}^2, \ \textit{on } \widehat{B}(0, C_{18}\mathcal{U}(t)), \quad C_{17} \leq t < +\infty. \tag{46}$$

By assumption (2) the sectional curvature of $g_{ij}(x,0)$ is strictly positive on M. Thus from the result of D. Gromoll and W. Meyer [21], M is diffeomorphic to $\mathbb{R}^{2n}$. Using the result of R.E. Greene and H. Wu [19], we know that M is exhausted by a family of convex domains, M is a Stein manifold. Namely, there exists a family of domains $\Omega_k \subset M$ for $k = 1, 2, 3, \ldots$ such that Ω_k is convex with respect to the metric ds_0^2 for every $k \geq 1$ and satisfies

$$\text{(47)} \qquad \text{(i)} \quad \overline{\Omega}_k \subset \Omega_{k+1}, \quad k = 1, 2, 3, \ldots,$$

$$\text{(48)} \qquad \text{(ii)} \quad B_0(x_0, 4k) \subset \Omega_k \subset B_0(x_0, \rho_k), \quad k = 1, 2, 3, \ldots,$$

where $\rho_1 < \rho_2 < \rho_3 < \ldots$ is a sequence of increasing positive numbers.

Since $\mathcal{U}(t) \to +\infty$ as $t \to +\infty$, for any integer $k \geq 1$, we can find a number t_k such that

$$\text{(49)} \qquad \text{(i)} \quad C_{17} \leq t_k < t_{k+1}, \quad k = 1, 2, 3, \ldots,$$

$$\text{(50)} \qquad \text{(ii)} \quad C_{18}\mathcal{U}(t) \geq 4\rho_k + 4, \ t_k \leq t < +\infty, \quad k = 1, 2, 3, \ldots.$$

For any $t \in [t_1, \infty)$, from Lemma 9.2 it follows that Φ_t is a holomorphic map from $\widehat{B}(0, C_{18}\mathcal{U}(t))$ to M, which is nonsingular at every point $z \in \widehat{B}(0, C_{18}\mathcal{U}(t))$. Thus Φ_t is a holomorphic covering map and is locally biholomorphic on $\widehat{B}(0, C_{18}\mathcal{U}(t))$. By (45), (46) and (50), there exist a small neighborhood W_t of x_0 on M and a biholomorphic map φ_t from W_t to $\varphi_t(W_t) \subset \mathbb{C}^n$ such that

$$\text{(51)} \qquad \text{(i)} \quad \Phi_t(\varphi_t(x)) \equiv x, \quad x \in W_t,$$

$$\text{(52)} \qquad \text{(ii)} \quad |\varphi_t(x_0)| \leq 2.$$

Since Φ_t depend on t smoothly, we can choose φ_t such that φ_t depend on t smoothly. For any integer $k \geq 1$ and any $t \in [t_k, \infty)$, from (48) and Lemma 9.1 (D) we have

$$\text{(53)} \qquad x_0 \in \Omega_k \subset B_0(x_0, \rho_k) \subset B_t(x_0, \rho_k).$$

Since Ω_k is convex with respect to the metric ds_0^2, Ω_k is simply connected. Since Φ_t is locally biholomorphic on $\widehat{B}(0, C_{18}\mathcal{U}(t))$, from (46), (49), (50), (51), (52) and (53) we know that for every $t \in [t_k, \infty)$, there is a unique biholomorphic map $\varphi_{k,t}$ from Ω_k to $\varphi_{k,t}(\Omega_k) \subset \mathbb{C}^n$ such that

$$\text{(54)} \qquad \text{(i)} \quad \Phi_t(\varphi_{k,t}(x)) \equiv x, \quad x \in \Omega_k,$$

$$\text{(55)} \qquad \text{(ii)} \quad |\varphi_{k,t}(x)| \leq 2\rho_k + 2, \quad x \in \Omega_k,$$

$$\text{(56)} \qquad \text{(iii)} \quad \varphi_{k,t}(x) = \varphi_t(x), \quad x \in \Omega_k \cap W_t.$$

Since φ_t depend on t smoothly, we know that $\varphi_{k,t}$ depend on t smoothly for $t_k \leq t < +\infty$. By the uniqueness of holomorphic extension and (56) we get

$$\varphi_{m,t}(x) = \varphi_{k,t}(x),\ x \in \Omega_k,\ m \geq k \geq 1,\ t_m \leq t < +\infty. \tag{57}$$

Lemma 9.3. *For any integer $k \geq 1$ and any $t_k \leq t < +\infty$, $\varphi_{k,t}(\Omega_k)$ is a bounded domain and is Runge in $\mathbb{C}^n$.*

Proof. By (55), $\varphi_{k,t}(\Omega_k)$ is a bounded domain in $\mathbb{C}^n$. A domain $\Omega \subset \mathbb{C}^n$ (not necessarily pseudoconvex) is said to be Runge in $\mathbb{C}^n$ if every holomorphic function on Ω can be approximated by entire functions, uniformly on compact sets in Ω. Suppose $f(z)$ is a holomorphic function on $\varphi_{k,t}(\Omega_k)$, we want to prove that $f(z)$ can be approximated uniformly on compact subsets by entire functions. Since $\varphi_{k,t}$ is biholomorphic in Ω_k, $f \circ \varphi_{k,t}$ is a holomorphic function in Ω_k. Since Ω_k is a convex domain in the Stein manifold (M, ds_0^2), using the $\overline{\partial}$ theory appeared in [25] we know that $f \circ \varphi_{k,t}$ can be approximated uniformly on compact subsets of Ω_k by holomorphic functions $h(x)$ defined on M. Thus from (50), (54) and (55) it follows that f can be approximated uniformly on compact subsets of $\varphi_{k,t}(\Omega_k)$ by holomorphic functions $h \circ \Phi_t$ defined on $\widehat{B}(0, 4\rho_k + 4)$. Since $\widehat{B}(0, 4\rho_k + 4)$ is a standard ball in $\mathbb{C}^n$, using the $\overline{\partial}$ theory appeared in [25] again we know that those holomorphic functions $h \circ \Phi_t$ can be approximated uniformly on compact subsets of $\widehat{B}(0, 4\rho_k + 4)$ by entire functions. Therefore f can be approximated uniformly on compact subsets of $\varphi_{k,t}(\Omega_k)$ by entire functions. Thus $\varphi_{k,t}(\Omega_k)$ is Runge in $\mathbb{C}^n$. q.e.d.

For any integer $k \geq 1$, we have already constructed biholomorphic maps $\varphi_{k,t}$ from Ω_k into $\mathbb{C}^n$. Now we want to construct the global biholomorphic map from M into $\mathbb{C}^n$. To do this we use the results of Andersen–Lempert [1] and Forstneric–Rosay [16] in 1992 and 1993. In their papers [1] and [16] some approximations of biholomorphic maps by automorphisms of $\mathbb{C}^n$ were obtained. By (48), $B_0(x_0, 4) \subset \Omega_1$. Since $\{\varphi_{1,t} | t_1 \leq t \leq t_2\}$ is a family of biholomorphic maps from Ω_1 into $\mathbb{C}^n$ which depends on t smoothly, thus there exists a constant $a_1 > 0$ such that

$$|\varphi_{1,t}(x) - \varphi_{1,t}(y)| \geq a_1 \gamma_0(x, y),\ t_1 \leq t \leq t_2,\ x, y \in \overline{B_0(x_0, 2)}, \tag{58}$$

where $\gamma_0(x, y)$ is the distance between x and y with respect to the metric ds_0^2. We define

$$f_t(z) = \varphi_{1,t}(\varphi_{1,t_2}^{-1}(z)),\ z \in \varphi_{1,t_2}(\Omega_1),\ t_1 \leq t \leq t_2. \tag{59}$$

Then $\{f_t | t_1 \le t \le t_2\}$ is a family of biholomorphic maps from $\varphi_{1,t_2}(\Omega_1)$ into $\mathbb{C}^n$, which depends on t smoothly and satisfies

$$f_{t_2}(z) = z, \quad z \in \varphi_{1,t_2}(\Omega_1), \tag{60 (i)}$$

$$f_t(\varphi_{1,t_2}(\Omega_1)) = \varphi_{1,t}(\Omega_1), \quad t_1 \le t \le t_2. \tag{61 (ii)}$$

For any $t_1 \le t \le t_2$, by Lemma 9.3, $\varphi_{1,t}(\Omega_1)$ is a bounded domain and is Runge in $\mathbb{C}^n$. Using the results of Andersen–Lempert [1] and Forstneric–Rosay [16] we know that $\{f_t | t_1 \le t \le t_2\}$ can be approximated uniformly on compact subsets of $\varphi_{1,t_2}(\Omega_1)$ by families $\{h_t | t_1 \le t \le t_2\}$ of automorphisms h_t of $\mathbb{C}^n$, which depend on t continuously and piecewise smoothly, and satisfy

$$h_{t_2}(z) = z, \quad z \in \mathbb{C}^n. \tag{62}$$

Thus $\{\varphi_{1,t} | t_1 \le t \le t_2\}$ can be approximated uniformly on compact subsets of Ω_1 by families $\{h_t \circ \varphi_{2,t_2} | t_1 \le t \le t_2\}$ of biholomorphic maps $h_t \circ \varphi_{2,t_2}$ from Ω_2 into $\mathbb{C}^n$, which depend on t continuously and piecewise smoothly, and satisfy

$$h_{t_2} \circ \varphi_{2,t_2}(x) = \varphi_{2,t_2}(x), \; x \in \Omega_2. \tag{63}$$

The approximation of $\varphi_{1,t}$ by $h_t \circ \varphi_{2,t_2}$ comes from (57) and (59). Since $B_0(x_0, 4) \subset \Omega_1$, using the derivative estimate for holomorphic functions we see that if $\varphi_{1,t} - h_t \circ \varphi_{2,t_2}$ is very close to zero on $\overline{B_0(x_0, 3)}$, then the derivatives of $\varphi_{1,t} - h_t \circ \varphi_{2,t_2}$ are very close to zero on $\overline{B_0(x_0, 2)}$. Therefore there exists a family $\{\varphi_{2,t} | t_1 \le t \le t_2\}$ of biholomorphic maps $\varphi_{2,t}$ from Ω_2 into $\mathbb{C}^n$ which depends on t continuously and piecewise smoothly such that

(64)

$$\text{(i)} \quad |\varphi_{2,t}(x) - \varphi_{1,t}(x)| \le \frac{1}{2}, \; x \in \overline{B_0(x_0, 2)}, \; t_1 \le t \le t_2,$$

(65)

$$\text{(ii)} \quad |\tilde{\nabla}[\varphi_{2,t}(x) - \varphi_{1,t}(x)]| \le \frac{1}{8} a_1, \; x \in \overline{B_0(x_0, 2)}, \; t_1 \le t \le t_2,$$

where $\tilde{\nabla}$ denote the covariant derivatives with respect to the metric ds_0^2. For any two points $x, y \in \overline{B_0(x_0, 1)}$, we can find a geodesic Λ from x to y such that the length of Λ is equal to $\gamma_0(x, y) \le 2$. Thus $\Lambda \subset \overline{B_0(x_0, 2)}$, which together with (65) implies

$$\begin{aligned} &|[\varphi_{2,t}(x) - \varphi_{1,t}(x)] - [\varphi_{2,t}(y) - \varphi_{1,t}(y)]| \\ &\le \frac{1}{8} a_1 \gamma_0(x, y), \quad x, y \in \overline{B_0(x_0, 1)}, \; t_1 \le t \le t_2. \end{aligned} \tag{66}$$

Combining (58) and (66) we get

$$|\varphi_{2,t}(x) - \varphi_{2,t}(y)| \geq \frac{7}{8} a_1 \gamma_0(x,y),$$
$$x, y \in \overline{B_0(x_0,1)},\ t_1 \leq t \leq t_2. \tag{67}$$

By the construction,

$$\varphi_{2,t}(x) = h_t(\varphi_{2,t_2}(x)), \quad x \in \Omega_2,\ t_1 \leq t \leq t_2, \tag{68}$$

where h_t are automorphisms of $\mathbb{C}^n$. Thus

$$\varphi_{2,t}(\Omega_2) = h_t(\varphi_{2,t_2}(\Omega_2)),\ t_1 \leq t \leq t_2. \tag{69}$$

By Lemma 9.3, $\varphi_{2,t_2}(\Omega_2)$ is a bounded domain and is Runge in $\mathbb{C}^n$. Thus for any $t_1 \leq t \leq t_2$, $\varphi_{2,t}(\Omega_2)$ is a bounded domain and is Runge in $\mathbb{C}^n$. From (63) and (68) it follows that $\{\varphi_{2,t} | t_1 \leq t < +\infty\}$ is a family of biholomorphic maps $\varphi_{2,t}$ from Ω_2 into $\mathbb{C}^n$, which depends on t continuously and piecewise smoothly. By (69) and Lemma 9.3 we know that for any $t_1 \leq t < +\infty$, $\varphi_{2,t}(\Omega_2)$ is a bounded domain and is Runge in $\mathbb{C}^n$.

Now we repeat the process. For any integer $k \geq 2$, using the results in [1] and [16] we can find a family $\{\varphi_{k,t} | t_1 \leq t \leq t_k\}$ of biholomorphic maps $\varphi_{k,t}$ from Ω_k into $\mathbb{C}^n$ which depends on t continuously and piecewise smoothly such that

(70)

(i) $\{\varphi_{k,t} | t_1 \leq t < +\infty\}$ depends on t continuously and piecewise smoothly,

(ii) For any $t_1 \leq t < +\infty$, $\varphi_{k,t}(\Omega_k)$ is a bounded domain and is Runge in $\mathbb{C}^n$,

(iii) $|\varphi_{k,t}(x) - \varphi_{k-1,t}(x)| \leq \left(\frac{1}{2}\right)^{k-1},\ x \in \overline{B_0(x_0, 2k-2)},$
$t_1 \leq t \leq t_k,$

$$\text{(iv)} \quad |[\varphi_{k,t}(x) - \varphi_{k-1,t}(x)] - [\varphi_{k,t}(y) - \varphi_{k-1,t}(y)]|$$

$$\leq \left(\frac{1}{2}\right)^{k+1} a_{k-1}\gamma_0(x,y), \tag{71}$$

$x, y \in \overline{B_0(x_0, k-1)},\ t_1 \leq t \leq t_k,$

where a_{k-1} is a constant such that

$$0 < a_{k-1} \leq a_{k-2} \leq \ldots \leq a_2 \leq a_1, \tag{72}$$

$$|\varphi_{k-1,t}(x) - \varphi_{k-1,t}(y)| \geq a_{k-1}\gamma_0(x,y), \tag{73}$$

$$x, y \in \overline{B_0(x_0, 2k-2)},\ t_1 \leq t \leq t_k.$$

By (71), (72) and (73) we get

$$|\varphi_{m,t}(x) - \varphi_{m,t}(y)| \geq \frac{3}{4} a_k \gamma_0(x,y),$$

$$x,\ y \in \overline{B_0(x_0,k)}, \quad t_1 \leq t \leq t_{k+1},\ m \geq k \geq 1. \tag{74}$$

Using (70) we obtain

$$|\varphi_{m,t}(x) - \varphi_{k,t}(x)| \leq \left(\frac{1}{2}\right)^{k-1},\ m \geq k \geq 1,$$

$$x \in \overline{B_0(x_0,2k)}, \quad t_1 \leq t \leq t_{k+1}. \tag{75}$$

From (75) it follows that $\{\varphi_{k,t_1} | k = 1,2,3,\ldots\}$ converges uniformly on compact subsets of M to a holomorphic map φ from M into $\mathbb{C}^n$:

$$\varphi(x) = \lim_{k \to +\infty} \varphi_{k,t_1}(x), \quad x \in M. \tag{76}$$

By (74) we have

$$|\varphi(x) - \varphi(y)| \geq \frac{3}{4} a_k \gamma_0(x,y),\ k \geq 1,\ x,\ y \in \overline{B_0(x_0,k)}. \tag{77}$$

Thus $\varphi : M \to \varphi(M) \subset \mathbb{C}^n$ is a biholomorphic map since $a_k > 0$ for any $k \geq 1$. Since M is a Stein manifold by the result of Greene–Wu [19], $\varphi(M)$ is a pseudoconvex domain in $\mathbb{C}^n$. Hence the proof of Theorem 1.2 is complete.

References

[1] E. Andersen & L. Lempert, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$, Invent. Math. **110** (1992) 371–388.

[2] A. Andreotti & E. Vesentini, *Carleman estimates for the Laplace–Beltrami equations on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. **25** (1965) 81–130.

[3] S. Bando, *On three-dimensional compact Kähler manifolds of nonnegative bisectional curvature*, J. Differential Geom. **19** (1984) 283–297.

[4] M. Berger, *Sur les groupes d'holonomie homogénes des variétés a connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955) 279–330.

[5] R. L. Bishop & R. J. Crittenden, *Geometry of Manifolds*, Academic Press, New York, 1964.

[6] S. Bochner & W. T. Martin, *Several Complex Variables*, Princeton University Press, Princeton, NJ, 1948.

[7] E. Calabi, *The space of Kähler metrics*, Proc. Internat. Congress Math., Amsterdam **2** (1954) 206–207.

[8] ———, *Improper affine hypersphere and generalization of a theorem of K. Jörgens*, Mich. Math. J. **5** (1958) 105–126.

[9] H. D. Cao, *Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985) 359–372.

[10] ———, *On Harnack's inequalities for the Kähler–Ricci flow*, Invent. Math. **109** (1992) 247–263.

[11] J. Cheeger & D. Ebin, *Comparison Theorems in Riemannian Geometry*, North–Holland Math. Library, North–Holland, Amsterdam, 1975.

[12] S. Y. Cheng & S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975) 333–354.

[13] C. Croke, *Some isoperimetric inequalities and consequences*, Ann. Sci. École Norm. Sup. **13** (1980).

[14] D. M. DeTurck, *Deforming metrics in direction of their Ricci tensors*, J. Differential Geom. **18** (1983) 157–162.

[15] P. G. Dixon & J. Esterle, *Michael's problem and the Poincaré–Fatou–Bieberbach phenomenon*, Bull. Amer. Math. Soc. **15** (1986) 127–187.

[16] F. Forstneric & J. P. Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbb{C}^n$, Invent. Math. **112** (1993) 323–349.

[17] L. W. Green, *A theorem of E. Hopf*, Michigan Math. J. **5** (1958) 31–34.

[18] R.E. Greene & H. Wu, *Analysis on noncompact Kähler manifolds*, Proc. Symp. Pure Math., Amer. Math. Soc. **30** (1977).

[19] ———, C^∞*-convex functions and manifolds of positive curvature*, Acta Math. **137** (1976) 209–245.

[20] ______, *Function theory on manifolds which possess a pole*, Lecture Notes in Math. Vol. 669, Springer, Berlin, 1979.

[21] D. Gromoll & W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. **90** (1969) 75–90.

[22] R. S. Hamilton, *Three–manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982) 255–306.

[23] ______, *Four–manifolds with positive curvature operator*, J. Differential Geom. **24** (1986) 153–179.

[24] ______, *The Harnack estimate for the Ricci flow*, J. Differential Geom. **37** (1993) 225–243.

[25] L. Hörmander , *An Introduction to Complex Analysis in Several Variables*, D. Van Nostrand, Princeton, New Jersey, 1966.

[26] ______, *L^2 estimate and existence theorems for the $\overline{\partial}$ operator*, Acta Math. **113** (1965) 89–152.

[27] A. Howard, B. Smyth & H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature. I*, Acta Math. **147** (1981) 51–56.

[28] G. Huisken, *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geom. **21** (1985) 47–62.

[29] J. J. Kohn & L. Nirenberg, *Non–coercive boundary problems*, Comm. Pure Appl. Math. **18** (1965) 443–492.

[30] P. Li & S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986) 153–201.

[31] N. Mok, Y. T. Siu & S. T. Yau, *The Poincaré–Lelong equation on complete Kähler manifolds*, Compositio Math. **44** (1981) 183–218.

[32] N. Mok, *An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties*, Bull. Soc. Math. France **112** (1984) 197–258.

[33] ______, *The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature*, J. Differential Geom. **27** (1988) 179–214.

[34] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. **110** (1979) 593–606.

[35] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961) 577–591.

[36] ______, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964) 101–134.

[37] M. H. Protter & H. F. Weinberger, *Maximum principles in differential equations*, Prentice Hall, Englewood Cliffs, NJ, 1967.

[38] J. P. Rosay & W. Rudin, *Holomorphic maps from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **310** (1988) 47–86.

[39] R. Schoen & S. T. Yau, *Differential Geometry*, Academia Sinica Press, Beijing, 1988.

[40] W. X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989) 223–301.

[41] ———, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. **30** (1989) 303–394.

[42] ———, *Complete noncompact three–manifolds with nonnegative Ricci curvature*, J. Differential Geom. **29** (1989) 353–360.

[43] ———, *Ricci deformation of the metric on complete noncompact Kähler manifolds*, Ph.D. thesis, Harvard University, 1990.

[44] ———, *Complete noncompact Kähler manifolds with positive holomorphic bisectional curvature*, Bull. Amer. Math. Soc. **23** (1990) 437–440.

[45] J. Simons, *On the transitivity of holnomy systems*, Ann. of Math. **76** (1962) 213–234.

[46] Y.T. Siu & S.T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980) 189–204.

[47] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990) 101–172.

[48] S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978) 339–411.

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